

ON STOCHASTIC EVOLUTION EQUATIONS FOR NONLINEAR BIPOLAR FLUIDS: WELL-POSEDNESS AND SOME PROPERTIES OF THE SOLUTION

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ABSTRACT. We investigate the stochastic evolution equations describing the motion of a Non-Newtonian fluids excited by multiplicative noise of Lévy type. We show that the system we consider has a unique global strong solution. We also give some results concerning the properties of the solution. Mainly we prove that the unique solution satisfies the Markov-Feller property. This enables us to prove by means of some results from ergodic theory that the semigroup associated to the unique solution admits at least an invariant measure which is ergodic and tight on a subspace of the Lebesgue space $L^2(\mathcal{O})$.

1. INTRODUCTION

Turbulence in Hydrodynamics is one of the most fascinating and difficult problems in Mathematics and in applied sciences in general. Many scientists believe that Newtonian law or the Navier-Stokes Equations (NSE for short) can accurately describe the most intricate complexities of turbulence in fluids flows. However, there are mainly two major obstacles for the mathematical study of turbulent flows. First, it is well known that the question of whether the three dimensional NSE admits or not a unique weak solution for all time still remains open. As it is not always easy to prove the existence of a global attractor in the case of lack of uniqueness of solution, this becomes a daunting obstacle for the investigation of the long-time behaviour of the Navier-Stokes equations which is very important for a better understanding of turbulence and some physical features of the fluids. We refer, for instance, to [3], [21], [22], [50], and [56] for some results in this direction. Second, there are a lot of fluid models exhibiting turbulent behavior that cannot be described by the Navier-Stokes equations. To overcome these problems one generally has to use other model of fluids or some regularizations, which might be of mathematical nature, of the Newtonian law. This has motivated many scientists to consider fluids such that their stress tensor is a nonlinear functions of the strain rate. This class of fluids forms the family of Non-Newtonian fluids. One examples of such fluids is the nonlinear Bipolar fluids which are themselves contained in the class of multipolar fluids. The theory of viscous multipolar fluids was initiated by Necas and Silhavy [42], and developed later on in numerous work of prominent scientists such as Necas, Novotny and Silhavy [43], Bellout, Bloom and Necas [4]. Although Bipolar fluids resembles to the models that Ladyzhenskaya considered in [37] and [38] they differ in two aspects. First both bipolar fluids and Ladyzhenskaya allow for a nonlinear velocity dependent viscosity, but in contrast to Bipolar fluids the Ladyzhenskaya models do not incorporate a higher-order velocity gradients. Second, in contrary to the Ladyzhenskaya models the theory of multipolar fluids is compatible with the basic principles of thermodynamics such as the Clausius-Duhem inequality and the principle of frame indifference. Moreover, results up to date indicate that the theory of multipolar fluids may lead to a better understanding of hydrodynamic turbulence (see for example [7]).

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Around the 70s Bensoussan and Temam [10] started the investigation of stochastic version of dynamical equations for Newtonian turbulent fluids. The Stochastic Partial Differential Equations they analyzed are obtained by adding noise terms to the deterministic NSE. This approach is basically motivated by Reynolds' work which stipulates that the velocity of a fluid particle in turbulent regime is composed of slow (deterministic) and fast (stochastic) components. While this belief was based on empirical and experimental data, Rozovskii and Mikulevicius were able to derive the models rigorously in their recent work [41], thereby confirming the importance of this approach in hydrodynamic turbulence. It is also pointed out in some recent articles like [31] and [36]) that some rigorous information on questions in Turbulence might be obtained from stochastic versions of the equations of fluid dynamics. Since the pioneering work of Bensoussan and Temam [10] on stochastic Navier-Stokes equations, stochastic models for Newtonian fluid dynamics and SPDEs in general have been the object of intense investigations which have generated several important results. We refer, for instance, to [1], [9],[12], [13], [16], [15], [17], [18], [19], [24],[25], [26], [27],[41], [44], [45],[53], [52], [54]. However, there are only very few results for the dynamical behaviour of stochastic models for Non-Newtonian fluids (see [34], [46], [47], [48], [29]).

In this paper, we are interested in the Lévy driven SPDEs for the nonlinear Bipolar fluids. More precisely, let $d = 2, 3$, and $\mathcal{O} \subset \mathbb{R}^d$ be a smooth bounded open domain, we consider

$$\begin{cases} d\mathbf{u} + [\mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \mathbf{T}(\mathcal{E}(\mathbf{u})) + \nabla \pi] dt = \int_{\mathcal{Z}} \sigma(t, \mathbf{u}, z) \tilde{\eta}(dz, dt), & x \in \mathcal{O}, t \in (0, T], \\ \mathbf{u}(x, 0) = \mathbf{u}_0, & x \in \mathcal{O}, \\ \nabla \cdot \mathbf{u} = 0, & x \in \mathcal{O}, t \in [0, T], \\ \mathbf{u}(x, t) = \tau_{ijl} n_j n_l = 0, & x \in \partial\mathcal{O}, t \in (0, T) \end{cases} \quad (1)$$

where \mathbf{u} is the velocity of the fluids, π its pressure, \mathbf{n} denotes the normal exterior to the boundary and

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i} \right), |\mathcal{E}(\mathbf{u})|^2 = \sum_{i,j=1}^n |\mathcal{E}_{ij}(\mathbf{u})|^2, \\ \mathbf{T}(\mathcal{E}(\mathbf{u})) = 2\kappa_0(\kappa + |\mathcal{E}(\mathbf{u})|^2)^{\frac{p-2}{2}} \mathcal{E}(\mathbf{u}) - 2\kappa_1 \Delta \mathcal{E}(\mathbf{u}).$$

The quantities κ_0 , κ_1 and κ denote positive constants. Here $\tilde{\eta}$ is a compensated Poisson random measure defined on a prescribed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking its values in a separable Hilbert space H to be defined later. The system (1) describes the equations of motion of isothermal incompressible nonlinear bipolar fluids excited by random forces.

For $p = 2$, $\kappa_1 = 0$, $\sigma \equiv 0$, (1) is the Navier-Stokes equations which has been extensively studied (see, for instance, [55]). If $1 < p < 2$ then the fluid is shear thinning, and it is shear thickening when $2 < p$. The problem (1) is as interesting as the Navier-Stokes equations. It contains two nonlinear terms which makes the problem as difficult as any nonlinear evolution equations. During the last two decades, the deterministic version of (1) has been the object of intense mathematical investigation which has generated several important results. We refer to [5], [6], [8], [39], [40] for relevant examples. Despite these numerous results there are still a lot of open problems related to the mathematical theory of multipolar fluids. Some examples are the existence of weak solution for all values of p , the uniqueness of such weak solutions and many more. We refer, for instance, to [6], [32] and [40] for some discussions about these challenges.

For $1 < p$ and the noise is replaced by a cylindrical Wiener process, the existence of martingale and stationary solution of (1) was established in [34]. Razafimandimby and Sango studied the exponential stability and some stabilization of (1) with $1 < p \leq 2$ and with a Wiener noise. It seems that this article is the first work studying the Lévy driven SPDEs (1). Our first main goal is to prove the existence and uniqueness of strong solution which should be understood in the sense of stochastic differential equations. To achieve this goal we mainly follow the idea initially developed by Breckner in [12] (see also [11]) and used later in many articles such as [18], [26], [49]. This method is based on Galerkin approximation and it allows to prove that the

whole sequence of the Galerkin approximation converges in mean square to the exact solution. The second of the present paper is to give some partial results concerning the properties of the solution. We concentrate on proving that the solution satisfies the Markov-Feller property which enables us to prove that (1) admits at least an invariant measure which is ergodic and tight on a subspace of the Lebesgue space $L^2(\mathcal{O})$. Unfortunately we could not proceed further and prove the uniqueness of the invariant measure. The investigation of the uniqueness of ergodic SPDEs driven by pure jump noise seems to be very difficult and out of reach of the most recent methods used to prove the uniqueness of invariant measure of SPDEs. We postpone this investigation in future work.

To close this introduction we give the outline of the article. In Section 2 we give most of the notations and necessary preliminary used throughout the work. By means of Galerkin approximation we show the existence of strong solution in Section 3. The pathwise uniqueness of the solution and the convergence of the whole sequence of Galerkin approximate solution to the exact solution are proved in Section 4. Section 5 is devoted to the investigation of some properties of the strong solution.

Notations:

By \mathbb{N} we denote the set of nonnegative integers, i.e. $\mathbb{N} = \{0, 1, 2, \dots\}$ and by $\bar{\mathbb{N}}$ we denote the set $\mathbb{N} \cup \{+\infty\}$. Whenever we speak about \mathbb{N} (or $\bar{\mathbb{N}}$)-valued measurable functions we implicitly assume that the set is equipped with the trivial σ -field $2^{\mathbb{N}}$ (or $2^{\bar{\mathbb{N}}}$). By \mathbb{R}_+ we will denote the interval $[0, \infty)$ and by \mathbb{R}_* the set $\mathbb{R} \setminus \{0\}$. If X is a topological space, then by $\mathcal{B}(X)$ we will denote the Borel σ -field on X . By λ_d we will denote the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, by λ the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

If (S, \mathcal{S}) is a measurable space then by $M(S)$ we denote the set of all real valued measures on (S, \mathcal{S}) , and by $\mathcal{M}(S)$ the σ -field on $M(S)$ generated by functions $i_B : M(S) \ni \mu \mapsto \mu(B) \in \mathbb{R}$, $B \in \mathcal{S}$. By $M_+(S)$ we denote the set of all non negative measures on S , and by $\mathcal{M}(S)$ the σ -field on $M_+(S)$ generated by functions $i_B : M_+(S) \ni \mu \mapsto \mu(B) \in \mathbb{R}_+$, $B \in \mathcal{S}$. Finally, by $M_I(S)$ we denote the family of all $\bar{\mathbb{N}}$ -valued measures on (S, \mathcal{S}) , and by $\mathcal{M}_I(S)$ the σ -field on $M_I(S)$ generated by functions $i_B : M(S) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{S}$. If (S, \mathcal{S}) is a measurable space then we will denote by $\mathcal{S} \otimes \mathcal{B}(\mathbb{R}_+)$ the product σ -field on $S \times \mathbb{R}_+$ and by $\nu \otimes \lambda$ the product measure of ν and the Lebesgue measure λ .

2. MATHEMATICAL SETTINGS OF THE PROBLEM (1)

Throughout this paper we mainly use the same notations as in [34]. By $L^q(\mathcal{O})$ we denote the Lebesgue space of q -th integrable functions with norm $\|\cdot\|_{L^q}$. For the particular case $q = 2$, we denote its norm by $\|\cdot\|$. For $q = \infty$ the norm is defined by $\|\mathbf{u}\|_{L^\infty} = \text{ess sup}_{x \in D} |\mathbf{u}(x)|$, where $|x|$ is the Euclidean norm of the vector $x \in \mathbb{R}^n$. The Sobolev space $\{\mathbf{u} \in L^q(\mathcal{O}) : D^k \mathbf{u} \in L^q(\mathcal{O}), k \leq \sigma\}$ with norm $\|\cdot\|_{q,\sigma}$ is denoted by $W^{q,\sigma}(\mathcal{O})$. $C(I, X)$ is the space of continuous functions from the interval $I = [0, T]$ to X , and $L^q(I, X)$ is the space of all measurable functions $u : [0, T] \rightarrow X$, with the norm defined by

$$\|\mathbf{u}\|_{L^q(I, X)}^q = \int_0^T \|\mathbf{u}(t)\|_X^q dt, \quad q \in [1, \infty),$$

and when $q = \infty$, $\|\mathbf{u}\|_{L^\infty(I, X)} = \text{ess sup}_{t \in [0, T]} \|\mathbf{u}(t)\|_X$.

The mathematical expectation associated to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is denoted by \mathbb{E} and as above we also define the space $L^q(\Omega, X)$.

We proceed with the definitions of some additional spaces frequently used in this work. We define a space of smooth functions with support strictly contained in \mathcal{O} and satisfying the divergence free condition:

$$\mathcal{V} = \{\mathbf{u} \in \mathcal{C}_c^\infty(\mathcal{O}) : \nabla \cdot \mathbf{u} = 0\}.$$

We denote by H the closure of \mathcal{V} with norm $\|\cdot\|$ in $L^2(\mathcal{O})$. It is a Hilbert space when equipped with the $L^2(\mathcal{O})$ -inner product (\cdot, \cdot) . \mathcal{H}^σ is the closure of \mathcal{V} in $W^{2,\sigma}(\mathcal{O})$ with the norm $\|\cdot\|_{2,\sigma}$.

We denote by $\|\mathbf{u}\|_\sigma$ the norm induced by $\|\mathbf{u}\|_{2,\sigma}$ on \mathcal{H}^σ . We denote by $\mathcal{H}^{-\sigma}$ the dual space of \mathcal{H}^σ ($\sigma \geq 1$) wrt the norm $\|\mathbf{u}\|_\sigma$. If $\sigma = 2$, then $V = \mathcal{H}^2$ and V^* is the dual space of V . The duality product between V and V^* is denoted by $\langle \cdot, \cdot \rangle$. It should be noted that V is not the usual space of divergence-free functions of $W^{2,1}(\mathcal{O})$ used for the Navier-Stokes equations. Here it is a space of divergence-free functions of $W^{2,2}(\mathcal{O})$. We assume throughout the paper that there exists a positive constant λ_1 such that the Poincaré inequalities type

$$\|\mathbf{u}\|_\sigma^2 \leq \frac{1}{\lambda_1} \|\mathbf{u}\|_{\sigma+1}^2, \forall \mathbf{u} \in \mathcal{H}^{\sigma+1}, \sigma \geq 0, \quad (2)$$

hold.

As mentioned in the introduction we will study a stochastic model for a nonlinear bipolar fluids excited by random forces. In the following lines we describe the forces acting on the fluids. Let (Z, \mathcal{Z}) be a separable metric space and let ν be a σ -finite positive measure on it. Suppose that $\mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration, and $\eta : \Omega \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{Z} \rightarrow \bar{\mathbb{N}}$ is a time homogeneous Poisson random measure with the intensity measure ν defined over the filtered probability space \mathfrak{P} . We will denote by $\tilde{\eta} = \eta - \gamma$ the compensated Poisson random measure associated to η where the compensator γ is given by

$$\mathcal{B}(\mathbb{R}_+) \times \mathcal{Z} \ni (A, I) \mapsto \gamma(A, I) = \nu(A)\lambda(I) \in \mathbb{R}_+.$$

We assume that $p \in (1, 2]$. This will be fixed in the whole section. Let $\mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H))$ be the class of all progressively measurable processes $\xi : \mathbb{R}_+ \times Z \times \Omega \rightarrow V$ satisfying the condition

$$\mathbb{E} \int_0^T \int_Z |\xi(r, z)|_H^2 \nu(dz) dr < \infty, \quad T > 0. \quad (3)$$

If $T > 0$, the class of all progressively measurable processes $\xi : [0, T] \times Z \times \Omega \rightarrow V$ satisfying the condition (3) just for this one T , will be denoted by $\mathcal{M}^2(0, T, L^2(Z, \nu, H))$. Also, let $\mathcal{M}_{step}^2(\mathbb{R}_+, L^2(Z, \nu, H))$ be the space of all processes $\xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H))$ such that

$$\xi(r) = \sum_{j=1}^n 1_{(t_{j-1}, t_j]}(r) \xi_j, \quad 0 \leq r,$$

where $\{0 = t_0 < t_1 < \dots < t_n < \infty\}$ is a partition of $[0, \infty)$, and for all j , ξ_j is an $\mathcal{F}_{t_{j-1}}$ measurable random variable. For any $\xi \in \mathcal{M}_{step}^2(\mathbb{R}_+, L^2(Z, \nu, H))$ we set

$$\tilde{I}(\xi) = \sum_{j=1}^n \int_Z \xi_j(z) \tilde{\eta}(dz, (t_{j-1}, t_j]). \quad (4)$$

Basically, this is the definition of stochastic integral of a random step process ξ with respect to the compound random Poisson measure $\tilde{\eta}$. The extension of this integral on $\mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H))$ is possible thanks to the following result which is taken from [14, Theorem C.1].

Theorem 2.1. *There exists a unique bounded linear operator*

$$I : \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu; H)) \rightarrow L^2(\Omega, \mathcal{F}; H)$$

such that for $\xi \in \mathcal{M}_{step}^2(\mathbb{R}_+, L^2(Z, \nu; H))$ we have $I(\xi) = \tilde{I}(\xi)$. In particular, there exists a constant $C = C(H)$ such that for any $\xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H))$,

$$\mathbb{E} \left| \int_0^t \int_Z \xi(r, z) \tilde{\eta}(dz, dr) \right|_H^2 \leq C \mathbb{E} \int_0^t \int_Z |\xi(r, z)|_H^2 \nu(dz) dr, \quad t \geq 0. \quad (5)$$

Moreover, for each $\xi \in \mathcal{M}^2(\mathbb{R}_+, L^2(Z, \nu, H))$, the process $I(1_{[0, t]} \xi)$, $t \geq 0$, is an H -valued càdlàg martingale. The process $1_{[0, t]} \xi$ is defined by $[1_{[0, t]} \xi](r, z, \omega) := 1_{[0, t]}(r) \xi(r, z, \omega)$, $t \geq 0$, $r \in \mathbb{R}_+$, $z \in Z$ and $\omega \in \Omega$.

As usual we will write

$$\int_0^t \int_Z \xi(r, z) \tilde{\eta}(dz, dr) := I(\xi)(t), \quad t \geq 0.$$

We will rewrite (1) in the following equivalent form

$$\begin{cases} d\mathbf{u} + [\kappa_1 \mathcal{A}\mathbf{u} + 2\kappa_0 \mathcal{A}_p \mathbf{u} + B(\mathbf{u}, \mathbf{u})] dt = \int_Z \sigma(t, \mathbf{u}, z) \tilde{\eta}(dz, dt), \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \quad (6)$$

Here the operator \mathcal{A} is defined through the relation

$$\langle \mathcal{A}\mathbf{u}, \mathbf{v} \rangle = a(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}} \frac{\partial \mathcal{E}_{ij}(\mathbf{u})}{\partial x_k} \frac{\partial \mathcal{E}_{ij}(\mathbf{v})}{\partial x_k} dx, \quad \mathbf{u} \in D(\mathcal{A}), \mathbf{v} \in V.$$

Here and after the summation over repeated indices is enforced.

Note that

$$D(\mathcal{A}) = \{\mathbf{u} \in V : \exists f \in H \subset V^* \text{ for which } a(\mathbf{u}, \mathbf{v}) = (f, \mathbf{v}), \forall \mathbf{v} \in V\},$$

$\mathcal{A} = \mathbf{P}\Delta^2$, where \mathbf{P} is the orthogonal projection defined on $L^2(\mathcal{O})$ onto H .

Remark 2.2. It is shown in [8] that there exist two positive constants k_1 and k_2 depending only on \mathcal{O} such that

$$k_1 \|\mathbf{u}\|_2^2 \leq \langle \mathcal{A}\mathbf{u}, \mathbf{u} \rangle \leq k_2 \|\mathbf{u}\|_2^2, \quad (7)$$

for any $\mathbf{u} \in V$. Thanks to this we will just write $\|\mathbf{u}\|_2^2$ in place of $\langle \mathcal{A}\mathbf{u}, \mathbf{u} \rangle$, $\mathbf{u} \in V$. Also, it is not difficult to see that \mathcal{A} is symmetric. This fact together with (7) yields that \mathcal{A} is self-adjoint.

We also introduce a bilinear form $B(\mathbf{u}, \mathbf{v}) : \mathcal{H}^1 \times \mathcal{H}^1 \rightarrow \mathcal{H}^{-1}$ as follows:

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\mathcal{O}} \mathbf{u}_i \frac{\partial \mathbf{v}_j}{\partial x_i} \mathbf{w}_j dx, \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{H}^1,$$

where $b(\cdot, \cdot, \cdot)$ is the well-known trilinear form used in the mathematical analysis of Navier-Stokes equations (see for instance [55]). The bilinear form $B(\cdot, \cdot)$ enjoys the following properties:

- for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{H}^1$, we have

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle = -\langle B(\mathbf{u}, \mathbf{w}), \mathbf{v} \rangle \text{ and } \langle B(\mathbf{u}, \mathbf{v}), \mathbf{v} \rangle = 0. \quad (8)$$

- There exists a constant C_0 such that

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle \leq C_0 \|\mathbf{u}\| \|\mathbf{v}\|_1 \|\mathbf{w}\|_2 \quad (9)$$

for any $\mathbf{u} \in \mathcal{H}^1, \mathbf{v} \in V, \mathbf{w} \in V$.

The inequality (9) can be proved by using Hölder's and Sobolev inequalities (see [55]). The nonlinear term $\mathcal{A}_p : V \rightarrow V^*$ is defined as follows:

$$(\mathcal{A}_p \mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}} \Gamma(\mathbf{u}) \mathcal{E}_{ij}(\mathbf{u}) \mathcal{E}_{ij}(\mathbf{v}) dx, \quad \mathbf{u}, \mathbf{v} \in V,$$

where $\Gamma(\mathbf{u}) = (\varkappa + |\mathcal{E}(\mathbf{u})|^2)^{\frac{p-2}{2}}$. Some of the properties of \mathcal{A}_p is given below.

Lemma 2.3. (i) *There exists a positive constant $C(\varkappa, p)$ such that*

$$\|\mathcal{A}_p \mathbf{u} - \mathcal{A}_p \mathbf{v}\|_{V^*} \leq C \|\mathbf{u} - \mathbf{v}\|_1, \quad \mathbf{u}, \mathbf{v} \in V. \quad (10)$$

(ii) *For any $\mathbf{u}, \mathbf{v} \in V$*

$$\langle \mathcal{A}_p \mathbf{u} - \mathcal{A}_p \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \geq 0. \quad (11)$$

To check the results in the above lemma we need to recall the following results whose proofs can be found in [32].

Lemma 2.4 (Korn's inequalities). *Let $1 < p < \infty$ and let $\mathcal{O} \subset \mathbb{R}^d$ be of class C^1 . Then there exist two positive constants $k_p^i = k^i(\mathcal{O}, p)$, $i = 1, 2$ such that*

$$k_p^1 \|\mathbf{u}\|_1 \leq \left(\int_{\mathcal{O}} |\mathcal{E}(\mathbf{u})|^2 dx \right)^{\frac{1}{2}} \leq k_p^2 \|\mathbf{u}\|_1,$$

for any $\mathbf{u} \in \mathcal{H}^1$.

Proof of Lemma 2.3. Let \mathbf{w} be any element of V . Let us set $\delta = \frac{|\mathbf{e}|}{\sqrt{\kappa}}$ and $\tilde{\kappa} = \max(\kappa^{\frac{p-2}{2}}, \kappa^{\frac{p-5}{2}})$. Let us first note that

$$\left| \frac{\partial \mathbf{T}}{\partial e_{ij}}(\mathbf{e}) \right| \leq \kappa^{\frac{p-2}{2}}(1 + \delta^2)^{\frac{p-2}{2}} + \kappa^{\frac{p-5}{2}}\delta(1 + \delta^2)^{\frac{p-4}{2}},$$

which implies that

$$\left| \frac{\partial \mathbf{T}}{\partial e_{ij}}(\mathbf{e}) \right| \leq 3\tilde{\kappa}(1 + \delta^2)^{\frac{p-2}{2}}.$$

Since $p \in (1, 2]$ we have that

$$\left| \frac{\partial \mathbf{T}}{\partial e_{ij}}(\mathbf{e}) \right| \leq 3\tilde{\kappa}. \quad (12)$$

Secondly, we have

$$\begin{aligned} \langle \mathbf{T}(\mathcal{E}(\mathbf{u})) - \mathbf{T}(\mathcal{E}(\mathbf{v})), \mathbf{w} \rangle &= 2\kappa_0 \int_{\mathcal{Q}} [\mathbf{T}(\mathcal{E}(\mathbf{u})) - \mathbf{T}(\mathcal{E}(\mathbf{v}))] \cdot \mathcal{E}(\mathbf{w}) dx, \\ &= 2\kappa_0 \int_{\mathcal{O}} \mathcal{E}(\mathbf{w}) \cdot \int_0^1 \frac{\partial \mathbf{T}(\mathcal{E}(\mathbf{v}) + s(\mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v})))}{\partial s} ds dx, \\ &= 2\kappa_0 \int_{\mathcal{O}} \mathcal{E}(\mathbf{w}) \cdot (\mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v})) \int_0^1 \frac{\partial \mathbf{T}(\mathcal{E}(\mathbf{v}) + s(\mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v})))}{\partial e_{ij}} ds dx. \end{aligned}$$

By using (12) in the last equation yields

$$|\langle \mathbf{T}(\mathcal{E}(\mathbf{u})) - \mathbf{T}(\mathcal{E}(\mathbf{v})), \mathbf{w} \rangle| \leq 6\kappa_0 \int_{\mathcal{O}} |\mathcal{E}(\mathbf{w})| \cdot |\mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v})| dx.$$

Invoking Hölder's and Korn's inequalities implies the existence of a positive constant K such that

$$|\langle \mathbf{T}(\mathcal{E}(\mathbf{u})) - \mathbf{T}(\mathcal{E}(\mathbf{v})), \mathbf{w} \rangle| \leq 6\tilde{\kappa}\kappa_0 K \|\mathbf{u} - \mathbf{v}\|_1 \|\mathbf{w}\|_1,$$

for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. We easily conclude from this the proof of (i).

It is known from [40] that for any $p \in (1, \infty)$ and for all $\mathbf{D}, \mathbf{E} \in \mathbb{R}_{sym}^{d \times d}$.

$$(\mathbf{T}(\mathbf{D}) - \mathbf{T}(\mathbf{E})) \cdot (\mathbf{D} - \mathbf{E}) \geq 0.$$

Here

$$\mathbb{R}_{sym}^{d \times d} = \{\mathbf{D} \in \mathbb{R}^{d \times d} : D_{ij} = D_{ji}, i, j = 1, 2, \dots, d\}.$$

Therefore, we see that for any $\mathbf{u}, \mathbf{v} \in V$

$$\langle \mathcal{A}_p \mathbf{u} - \mathcal{A}_p \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = \int_{\mathcal{Q}} [\mathbf{T}(\mathcal{E}(\mathbf{u})) - \mathbf{T}(\mathcal{E}(\mathbf{v}))] \cdot [\mathcal{E}(\mathbf{u}) - \mathcal{E}(\mathbf{v})] dx \geq 0,$$

which proves (ii). □

To close this section we introduce the main set of hypotheses used in this article. Throughout this work we suppose that we are given a function σ satisfying the following set of constraints:

Condition 1. *There exist nonnegative constants $\ell_0, \ell_1, \ell_2, \ell_3$ such that, for any $t \in [0, T]$ and all $\mathbf{u}, \mathbf{v} \in H$, we have*

- (1) $|\sigma(t, \mathbf{u})|_{L^2(Z, \nu; H)}^2 \leq \ell_0 + \ell_1 |\mathbf{u}|^2;$
- (2) $|\sigma(t, \mathbf{u}) - \sigma(t, \mathbf{v})|_{L^2(Z, \nu; H)}^2 \leq \ell_2 |\mathbf{u} - \mathbf{v}|^2.$
- (3) $|\sigma(t, \mathbf{u})|_{L^4(Z, \nu; H)}^4 \leq \ell_3 (1 + |\mathbf{u}|^4).$

3. EXISTENCE OF A STRONG SOLUTION

This section is devoted to show that (1) admits at least one strong solution. The proof is based on Galerkin approximation and idea borrowed from [12]. But before we proceed further we define explicitly what we mean by strong solution of (1) or (6).

Definition 3.1. Let (Z, \mathcal{Z}) be a separable metric space on which is defined a σ -finite measure ν and $\mathbf{u}_0 \in H$. A strong solution to the problem (6) is a stochastic process \mathbf{u} such that

(1) $\mathbf{u} = \{\mathbf{u}(t)\}_{t \geq 0}$ is a \mathbb{F} -progressively measurable process such that

$$\mathbb{E} \sup_{s \in [0, T]} |\mathbf{u}(s)|^4 + \mathbb{E} \int_0^T \|\mathbf{u}(t)\|_2^2 dt < \infty,$$

(2) the following holds

$$\begin{aligned} (\mathbf{u}(t), \mathbf{w}) = & (\mathbf{u}_0, \mathbf{w}) - \kappa_1 \int_0^t (\langle \mathcal{A}\mathbf{u}(s), \mathbf{w} \rangle - \langle B(\mathbf{u}(s), \mathbf{u}(s)), \mathbf{w} \rangle) ds \\ & - 2\kappa_0 \int_0^t \langle \mathcal{A}_p \mathbf{u}(s), \mathbf{w} \rangle ds + \int_0^t \int_Z (\sigma(s, \mathbf{u}(s), z), \mathbf{w}) \tilde{\eta}(dz, ds), \end{aligned} \quad (13)$$

for any $\mathbf{w} \in V$, for almost all $t \in [0, T]$ and \mathbb{P} -almost surely.

Theorem 3.2. *Let the set of constraints in Condition 1 be satisfied. Then for any $T > 0$ and initial value ξ with $\mathbb{E}|\xi|^4 < \infty$, there exists a solution $\mathbf{u} = \{\mathbf{u}(t) : 0 \leq t < \infty\}$ to problem (6) which satisfies*

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |\mathbf{u}(t)|^{2r} + \int_0^T \|\mathbf{u}(s)\|_2^2 |\mathbf{u}(s)|^{2r-2} ds \right) \leq C \left(\mathbb{E} |\xi|^{2r} + 1 \right), T \geq 0, \quad (14)$$

with $r = 1, 2$.

Before we prove this result let us recall an important statement which is borrowed from [23].

Lemma 3.3. *Let X, Y, I and ϕ be non-negative processes and Z be a non-negative integrable random variable. Assume that I is non-decreasing and that there exist non-negative constants $C, \alpha, \beta, \gamma, \delta$ and T satisfying first*

$$\int_0^T \phi(s) ds \leq C, \quad a.s., \quad 2\beta e^C \leq 1, \quad 2\delta e^C \leq \alpha,$$

and secondly for all $t \in [0, T]$ there exists a constant $\tilde{C} > 0$ such that

$$\begin{aligned} X(t) + \alpha Y(t) & \leq Z + \int_0^t \phi(r) X(r) dr + I(t), \quad a.s., \\ \mathbb{E} I(t) & \leq \beta \mathbb{E} X(t) + \gamma \int_0^t \mathbb{E} X(s) ds + \delta \mathbb{E} Y(t) + \tilde{C}. \end{aligned}$$

If $X \in L^\infty([0, T] \times \Omega)$, then we have

$$\mathbb{E} [X(t) + \alpha Y(t)] \leq 2 \exp(C + 2t\gamma e^C) (\mathbb{E} Z + \tilde{C}), \quad t \in [0, T].$$

The proof of Theorem 3.2 will be split into five steps.

A priori uniform estimates:

The operator \mathcal{A} is self-adjoint and it follows from Rellich's theorem that it is compact on H . Therefore, there exists a sequence of positive numbers $\{\tilde{\lambda}_i : i = 1, 2, 3, \dots\}$ and a family of smooth function $\{\phi_i : i = 1, 2, 3, \dots\}$ satisfying

$$\mathcal{A}\phi_i = \tilde{\lambda}_i \phi_i, \quad (15)$$

for any $i \in \mathbb{N}$. We can assume that the family $\{\phi_i : i = 1, 2, 3, \dots\}$ so that it will form an orthonormal basis of H which is orthogonal and dense in V .

Let Π_m denote the projection of V^* onto $H_m := \text{span}\{\phi_1, \dots, \phi_m\}$. That is

$$\Pi_m x = \sum_{i=1}^m \langle x, \phi_i \rangle \phi_i, x \in V^*.$$

Also, $\Pi_m|_H$ is the orthogonal projection of H onto H_m .

For every $m \in \mathbb{N}$, we consider the finite dimensional system of SDEs on H_m given by

$$\begin{aligned} d\mathbf{u}^m(t) &= \Pi_m F(\mathbf{u}^m(t)) dt + \int_Z \Pi_m \sigma(t, \mathbf{u}^m(t), z) \tilde{\eta}(dz, dt), \quad t \geq 0, \\ \mathbf{u}^m(0) &= \Pi_m \xi, \end{aligned} \quad (16)$$

where $F(\mathbf{u}^m(s)) = -A\mathbf{u}^m(s) - \mathcal{A}_p \mathbf{u}^m(s) + B(\mathbf{u}^m(s), \mathbf{u}^m(s))$. To shorten notation we set $B_m(\cdot, \cdot) = \Pi_m B(\cdot, \cdot)$ and $\sigma_m(\cdot, \cdot, \cdot) = \Pi_m \sigma(\cdot, \cdot, \cdot)$.

We note that since Π_m is a contraction of V^* , we infer from (8), (9) and point (2) of Condition 1 that F is locally Lipschitz and $\sigma_m := \Pi_m \sigma$ is globally Lipschitz. As we know from e.g. Albeverio, Brzeźniak and Wu [1], on the basis of Condition 1, equation (16) has a unique H_m -valued càdlàg local strong solution \mathbf{u}^m . The following proposition implies that it is in fact a global solution.

Proposition 3.4. *Let the assumptions be as in Theorem 3.2. Then there exists a constant $C > 0$ such that for $r = 1, 2$ we have*

$$\sup_m \mathbb{E} \left(\sup_{t \in [0, T]} |\mathbf{u}^m(t)|^{2r} + \int_0^T \|\mathbf{u}^m(s)\|_2^2 |\mathbf{u}^m(s)|^{2r-2} ds \right) \leq C \left(\mathbb{E} |\xi|^{2r} + 1 \right).$$

Proof of Proposition 3.4. As mentioned above, it follows from in [1, Theorem 2.8] that Equation (16) has a unique càdlàg local strong solution \mathbf{u}^m in H_m . That means that for any $m \in \mathbb{N}$ there exists a unique solution on a short interval interval $[0, T_m]$ satisfying

$$\mathbf{u}^m(t) = \Pi_m \xi + \int_0^t \Pi_m F(\mathbf{u}^m(s)) ds + \int_0^t \int_Z \sigma_m(s, \mathbf{u}^m(s), z) \tilde{\eta}(dz, ds), \quad t \in [0, T_m].$$

We begin by checking the estimate in the proposition with the case $r = 1$. We argue as in [1, Proof of Theorem 3.1]. Let $(\tau_M)_M$ be an increasing sequence of stopping times defined by

$$\tau_M = \inf\{t \in [0, T] : |\mathbf{u}^m(t)|^2 + \int_0^t \|\mathbf{u}^m(s)\|_2^2 ds \geq M^2\} \wedge T,$$

for any integer M . Throughout we fix $r \in [0, T]$ and $0 \leq t \leq r \wedge \tau_M$. Since we can identify the space H_m with \mathbb{R}^m then we can apply the finite dimensional Itô's formula (see, for example, [35, Chapter II, Theorem 5.1]) to the function $|\cdot|^{2r}$ and the process \mathbf{u}^m . This procedure along with (8) yields

$$\begin{aligned} |\mathbf{u}^m(t)|^2 &= |\Pi_m \xi|^2 - 2\kappa_1 \int_0^t \|\mathbf{u}^m(s)\|_2^2 ds - 2\kappa_0 \int_0^t \langle \mathcal{A}_p \mathbf{u}^m(s), \mathbf{u}^m(s) \rangle ds \\ &\quad + 2 \int_0^t \int_Z (\mathbf{u}^m(s-), \sigma_m(s, \mathbf{u}^m(s), z)) \tilde{\eta}(dz, ds) \\ &\quad + \int_0^t \int_Z \Psi(s, z) \eta(dz, ds), \end{aligned} \quad (17)$$

where

$$\Psi(s, z) = |\mathbf{u}^m(s-) + \sigma_m(s, \mathbf{u}^m(s), z)|^2 - |\mathbf{u}^m(s-)|^2 - (\mathbf{u}^m(s-), \sigma(s, \mathbf{u}^m(s), z)).$$

From the fact that $|x|^2 - |y|^2 + |x - y|^2 = 2\langle x - y, x \rangle$, $x, y \in H$ and (11), we derive from (17) that

$$\begin{aligned} |\mathbf{u}^m(t)|^2 + 2 \int_0^t \kappa_1 \|\mathbf{u}^m(s)\|_2^2 ds &\leq |\Pi_m \xi|^2 + 2 \int_0^t \int_Z (\mathbf{u}^m(s-), \sigma(s, \mathbf{u}^m(s), z)) \tilde{\eta}(dz, ds) \\ &\quad + \int_0^t \int_Z |\sigma(s, \mathbf{u}^m(s), z)|^2 \eta(dz, ds) \end{aligned} \quad (18)$$

for any $t \in [0, r \wedge \tau_M]$, $r \in [0, T]$. For any $r \in [0, T]$ and $t \in [0, r \wedge \tau_m]$ we define the following stochastic processes

$$\begin{aligned} X(t) &:= \sup_{0 \leq s \leq t} |\mathbf{u}^m(s)|^2; \\ Y(t) &:= \int_0^t \|\mathbf{u}^m(s)\|_2^2 ds; \\ I(t) &:= \sup_{0 \leq s \leq t} \left(2 \left| \int_0^s \int_Z (\mathbf{u}^m(\tau-), \sigma(\tau, \mathbf{u}^m(\tau), z)) \tilde{\eta}(dz, d\tau) \right| + \int_0^s \int_Z |\sigma(\tau, \mathbf{u}^m(\tau), z)|^2 \eta(dz, d\tau) \right), \\ &:= \sup_{s \in [0, t]} |I_1(s)| + I_2(t), \end{aligned}$$

where

$$I_1(t) = \int_0^t \int_Z (\sigma(s, \mathbf{u}^m(s-), z), \mathbf{u}^m(s-)) \tilde{\eta}(dz, ds),$$

and

$$I_2(t) = \sup_{0 \leq s \leq t} \int_0^s \int_Z |\sigma(\tau, \mathbf{u}^m(\tau-), z)|^2 \eta(dz, d\tau).$$

Since $I_1(t)$ is a local martingale we can apply Burkholder-Davis-Gundy's inequality and get

$$\mathbb{E} \sup_{s \in [0, r \wedge \tau_M]} |I_1(s)| \leq C \mathbb{E} \left(\int_0^{r \wedge \tau_m} \int_Z (\mathbf{u}^m(s-), \sigma(s, \mathbf{u}^m(s), z))^2 \nu(dz) ds \right)^{\frac{1}{2}}.$$

Thanks to Hölder's and Young's inequalities we have

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |I_1(s)| &\leq C \left[\varepsilon \mathbb{E} \sup_{s \in [0, t]} |\mathbf{u}^m(s)|^2 \right]^{\frac{1}{2}} \left[\frac{1}{\varepsilon} \mathbb{E} \int_0^t \int_Z |\sigma(s, \mathbf{u}^m(s), z)|^2 \nu(dz) ds \right]^{\frac{1}{2}} \\ &\leq C \varepsilon \mathbb{E} \sup_{s \in [0, t]} |\mathbf{u}^m(s)|^2 + \frac{C}{\varepsilon} \mathbb{E} \int_0^t \int_Z |\sigma(s, \mathbf{u}^m(s), z)|^2 \nu(dz) ds. \end{aligned}$$

Invoking item (2) of Condition 1 we see that

$$\mathbb{E} \sup_{s \in [0, t]} |I_1(s)| \leq C \varepsilon \mathbb{E} X(t) + \frac{C}{\varepsilon} \ell_0 t + \frac{C}{\varepsilon} \ell_1 \int_0^t \mathbb{E} X(s) ds. \quad (19)$$

Next, we will deal with the second term of $I(t)$. Taking into account that the process

$$\int_0^t \int_Z |\sigma_m(r, \mathbf{u}^m(r))|^2 \eta(dz, dr)$$

has only positive jumps, we obtain

$$\mathbb{E} I_2(t) \leq \mathbb{E} \int_0^t \int_Z |\sigma(s, \mathbf{u}^m, z)|^2 \nu(dz) ds.$$

Thanks to the item (1) of Condition 1 we see that

$$\mathbb{E} I_2(t) \leq \ell_0 t + \ell_1 \int_0^t \mathbb{E} X(s) ds. \quad (20)$$

Thanks to (18) along with (19) and (20) we apply Lemma 3.3 and derive that there exist a positive constant C such that

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} |\mathbf{u}^m(s)|^2 + 2\kappa_1 \int_0^t \|\mathbf{u}^m(s)\|_2 ds \right] \leq C(\mathbb{E}|\xi|^2 + 1),$$

for any $m \in \mathbb{N}$ and $t \in [0, r \wedge \tau_m]$, $r \in [0, T]$. We have just shown that

$$\mathbb{E} \left[\sup_{0 \leq s \leq t \wedge \tau_M} |\mathbf{u}^m(s)|^2 + 2\kappa_1 \int_0^t \|\mathbf{u}^m(s)\|_2 ds \right] \leq C(\mathbb{E}|\xi|^2 + 1) \forall t \in [0, T], \quad (21)$$

with which we can infer that

$$\mathbb{P}(\tau_M < t) \leq CM^{-2}, \forall t \in [0, T], \forall M > 0.$$

Hence, $\lim_{M \rightarrow \infty} \mathbb{P}(\tau_M < t) = 0$, for all $t \in [0, T]$. That is, $\tau_M \rightarrow \infty$ in probability. Therefore, there exists a subsequence τ_{M_k} such that $\tau_{M_k} \rightarrow \infty$, a.s. Since the sequence $(\tau_M)_M$ is increasing, we infer that $\tau_{M_k} \nearrow \infty$ a.s.. Now we use Fatou's lemma and pass to the limit in (21) and derive that

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} |\mathbf{u}^m(s)|^2 + 2\kappa_1 \int_0^t \|\mathbf{u}^m(s)\|_2 ds \right] \leq C(\mathbb{E}|\xi|^2 + 1).$$

The proposition is then proved for $r = 1$. Thus, it remains to show that it is true for the case $r = 2$. We again apply Itô's formula to obtain

$$\begin{aligned} |\mathbf{u}^m(t)|^{2r} &= 2r \int_0^t \int_Z |\mathbf{u}^m(s-)|^{2(r-1)} (\mathbf{u}^m(s-), \sigma_m(s, \mathbf{u}^m(s), z)) \tilde{\eta}(dz, ds) \\ &\quad + |\Pi_m \xi|^{2r} + 2r \int_0^t |\mathbf{u}^m(s)|^{2(r-1)} (F_m(\mathbf{u}^m(s)), \mathbf{u}^m(s)) ds \\ &\quad + \int_0^t \int_Z \Phi(s, z) \eta(dz, ds), \end{aligned} \quad (22)$$

where

$$\Phi(s, z) = |\mathbf{u}^m(s-) + \sigma_m(s, \mathbf{u}^m(s), z)|^{2r} - |\mathbf{u}^m(s-)|^{2r} - 2r |\mathbf{u}^m(s-)|^{2(r-1)} (\mathbf{u}^m(s-), \sigma(s, \mathbf{u}^m(s), z)).$$

Thanks to (8) and (11) the estimate (22) becomes

$$\begin{aligned} |\mathbf{u}^m(t)|^{2r} + 2r\kappa_1 \int_0^t |\mathbf{u}^m(s)|^{2(r-1)} \|\mathbf{u}^m(s)\|_2^2 ds &- |\Pi_m \xi|^{2r} - \int_0^t \int_Z \Phi(s, z) \eta(dz, ds) \\ &\leq 2r \int_0^t \int_Z |\mathbf{u}^m(s-)|^{2(r-1)} (\mathbf{u}^m(s-), \sigma_m(s, \mathbf{u}^m(s), z)) \tilde{\eta}(dz, ds). \end{aligned}$$

Taking the supremum over $[0, t]$ on both sides of the above estimate leads to

$$\sup_{s \in [0, t]} |\mathbf{u}^m(s)| + 2r\kappa_1 \int_0^t |\mathbf{u}^m(s)|^{2(r-1)} \|\mathbf{u}^m(s)\|_2^2 ds \leq |\Pi_m \xi|^{2r} + J(t), \quad (23)$$

where $J(t) = J_1(t) + J_2(t)$ with

$$\begin{aligned} J_1(t) &= 2r \sup_{s \in [0, t]} \left| \int_0^s \int_Z |\mathbf{u}^m(s-)|^{2(r-1)} (\mathbf{u}^m(s-), \sigma_m(s, \mathbf{u}^m(s), z)) \tilde{\eta}(dz, d\tau) \right|, \\ J_2(t) &= \sup_{s \in [0, t]} \left| \int_0^s \int_Z \Phi(\tau, z) \eta(dz, d\tau) \right|. \end{aligned}$$

First, we apply the Burkholder-Davis-Gundy inequality

$$\mathbb{E} J_1(t) \leq 2rC \mathbb{E} \left(\int_0^t \int_Z |\mathbf{u}^m(s)|^{4(q-1)} |\mathbf{u}^m(s)|^2 |\sigma_m(s, \mathbf{u}^m(s), z)|^2 \nu(dz) ds \right)^{\frac{1}{2}}.$$

Then using item (1) of Condition 1 and Hölder's inequality implies.

$$\begin{aligned} \mathbb{E}J_1(t) &\leq \left(\frac{1}{\varepsilon} \mathbb{E} \int_0^t |\mathbf{u}^m(s)|^{2r-2} \left(\ell_0 + \ell_1 |\mathbf{u}^m(s)|^2 \right) ds \right)^{\frac{1}{2}} \\ &\quad \times 2rC \left(\varepsilon \mathbb{E} \sup_{s \in [0,t]} |\mathbf{u}^m(s)|^{2r} \right)^{\frac{1}{2}}. \end{aligned}$$

Invoking Young's inequality yields

$$\mathbb{E}J_1(t) \leq \frac{2rC}{2} \varepsilon \mathbb{E} \sup_{s \in [0,t]} |\mathbf{u}^m(s)|^{2r} + \frac{rC\ell_0}{2\varepsilon} \mathbb{E} \int_0^t |\mathbf{u}^m(s)|^{2r-2} ds + \frac{2rC\ell_1}{2\varepsilon} \mathbb{E} \int_0^t |\mathbf{u}^m(s)|^{2r} ds.$$

Using the fact that for $r \geq 2$, $|x|^{2r-2} \leq 1 + |x|^{2r}$ we deduce from the last inequality that

$$\mathbb{E}J_1(t) \leq \frac{2rC\ell_0 T}{2\varepsilon} + \frac{2rC}{2} \varepsilon \mathbb{E} \sup_{s \in [0,t]} |\mathbf{u}^m(s)|^{2r} + \frac{2rC(\ell_0 + \ell_1)}{2\varepsilon} \mathbb{E} \int_0^t |\mathbf{u}^m(s)|^{2r} ds. \quad (24)$$

Now we deal with $J_2(t)$. First, note that

$$\mathbb{E}J_2(t) \leq \left(\frac{r^2 + r}{2} \right) \mathbb{E} \int_0^t \int_Z \left(|\mathbf{u}^m(s-)|^{2(r-1)} |\sigma_m(s, \mathbf{u}^m(s), z)|^2 + |\sigma_m(s, \mathbf{u}^m(s), z)|^{2r} \right) \nu(dz) ds, \quad (25)$$

where we have used the fact that

$$\left| |x+h|^{2r} - |x|^{2r} - 2r|x|^{2(r-1)}(x, h) \right| \leq \frac{r^2 + r}{2} (|x|^{2(r-1)}|h|^2 + |h|^{2r}), \quad (26)$$

for all $x, h \in H$. Let set $C_r = \frac{r^2+r}{2}$. Now thanks to items (1) and (3) of Condition 1 we derive from (25) that there exist positive constants ℓ_r and C_r such that

$$\begin{aligned} \mathbb{E}J_2(t) &\leq C_r \ell_r \mathbb{E} \int_0^t (1 + |\mathbf{u}^m(s)|^{2r} + |\mathbf{u}^m(s)|^{2(r-1)} \|\mathbf{u}^m(s)\|_2^{2r}) ds \\ &\leq C_r \ell_r T + C_r \ell_r \int_0^t |\mathbf{u}^m(s)|^{2r} ds + C_r \ell_r \int_0^t |\mathbf{u}^m(s)|^{2(r-1)} \|\mathbf{u}^m(s)\|_2^{2r} ds. \end{aligned} \quad (27)$$

Therefore we see from (24) and (27) that there exist positive constants C'_r , M_r , ℓ'_r , and L'_r such that

$$\mathbb{E}J(t) \leq C'_r T + M_r \varepsilon \mathbb{E} \sup_{s \in [0,t]} |\mathbf{u}^m(s)|^{2r} + \frac{\ell'_r}{2\varepsilon} \int_0^t \mathbb{E} |\mathbf{u}^m(s)|^{2r} ds + L'_r \mathbb{E} \int_0^t |\mathbf{u}^m(s)|^{2(r-1)} \|\mathbf{u}^m(s)\|_1^2 ds, \quad (28)$$

for any $m \in \mathbb{N}$ and $t \in [0, T]$. Let set

$$X(t) = \sup_{s \in [0,t]} |\mathbf{u}^m(s)|^{2r},$$

and

$$Y(t) = \int_0^t |\mathbf{u}^m(s)|^{2(r-1)} \|\mathbf{u}^m(s)\|_2^2 ds.$$

Thanks to (23), (28) and an appropriate choice of $\varepsilon > 0$ we find that $X(\cdot)$ and $Y(\cdot)$ verifies the conditions in Lemma 3.3. Therefore we infer the existence of a positive constant C such that

$$\mathbb{E} \sup_{s \in [0,t]} |\mathbf{u}^m(s)|^{2r} + C(q, \kappa_1) \int_0^t |\mathbf{u}^m(s)|^{2(r-1)} \|\mathbf{u}^m(s)\|_2^2 ds \leq C(\mathbb{E}|\xi|^{2r} + 1), r = 2,$$

for any $t \in [0, T]$ and $m \in \mathbb{N}$. This completes the proof of the proposition. \square

Passage to the limit:

To prove the existence of the solution of (6) we need to pass to the limit in the terms of (16) and in the estimate of Proposition 3.4. Before we do so we recall that there exists a constant $C > 0$ such that

$$\sup_{m \in \mathbb{N}} \left(\mathbb{E} \sup_{s \in [0, T]} |\mathbf{u}^m(s)|^{2r} + \mathbb{E} \int_0^T |\mathbf{u}^m(s)|^{2(r-1)} \|\mathbf{u}^m(s)\|_2^2 ds \right) < C. \quad (29)$$

$$(30)$$

We have the following weak compactness result.

Proposition 3.5. *We can extract from \mathbf{u}^m a subsequence which is not relabeled and there exists a stochastic process \mathbf{u} such that*

$$\mathbf{u}^m \rightharpoonup \mathbf{u} \text{ (weak star) in } L^4(\Omega; L^\infty([0, T]; H)), \quad (31)$$

$$\mathbf{u}^m \rightharpoonup \mathbf{u} \text{ in } L^2(\Omega \times [0, T]; V). \quad (32)$$

Moreover, there exists three elements $\mathbf{B}, \Sigma, \mathbf{A}$ such that

$$B_m(\mathbf{u}^m, \mathbf{u}^m) \rightharpoonup \mathbf{B} \text{ in } L^2(\Omega \times [0, T]; V^*), \quad (33)$$

$$\mathcal{A}_p \mathbf{u}^m \rightharpoonup \mathbf{A} \text{ in } L^2(\Omega \times [0, T]; V^*), \quad (34)$$

$$\sigma_m(t, \mathbf{u}^m, \cdot) \rightharpoonup \Sigma \text{ in } L^2(\Omega \times [0, T]; L^2(Z, \nu; H)). \quad (35)$$

Proof. Since $L^2(\Omega \times [0, T]; V)$ is a Hilbert space and $L^4(\Omega; L^\infty(0, T; H))$ is a Banach space and the dual of $L^{\frac{4}{3}}(\Omega; L^1(0, T; H))$, we easily infer from Banach-Alaoglu's theorem and the uniqueness of weak limit that there exist a subsequence of \mathbf{u}^m (which is denoted with the same fashion) and a stochastic process \mathbf{u} belonging to $L^4(\Omega; L^\infty(0, T; H)) \cap L^2(\Omega \times [0, T]; V)$ such that (31) and (32) hold true.

It remains to show (33)-(35). To prove (33) we first recall that there exists a positive constant C_0 such that

$$|\langle B_m(\Phi, \mathbf{v}), \mathbf{w} \rangle| \leq C_0 |\Phi| \|\mathbf{v}\|_1 \|\mathbf{w}\|_2,$$

for any $\Phi \in H$, $\mathbf{v} \in V$ and $\mathbf{w} \in V$. This inequality implies that

$$\mathbb{E} \int_0^T \|B_m(\mathbf{u}^m(s), \mathbf{u}^m(s))\|_{V^*}^2 ds \leq C \mathbb{E} \int_0^T |\mathbf{u}^m(s)|^2 \|\mathbf{u}^m(s)\|_1^2 ds,$$

from which and (29) we get that $B_m(\mathbf{u}^m, \mathbf{u}^m)$ is a bounded sequence in the Hilbert space $L^2(\Omega \times [0, T]; V^*)$. Thus, there exists an element of $L^2(\Omega \times [0, T]; V^*)$ that we denote by \mathbf{B} such that $B_m(\mathbf{u}^m, \mathbf{u}^m)$ converges weakly to \mathbf{B} in $L^2(\Omega \times [0, T]; V^*)$.

By invoking (10) and (29) we see that the following uniform estimate holds

$$\sup_{m \in \mathbb{N}} \mathbb{E} \int_0^T \|\mathcal{A}_p \mathbf{u}^m(s)\|_{V^*}^2 ds \leq C.$$

Therefore, the proof of (34) follows the same lines as for the proof of (33).

From point (1) of Condition 1 and estimate (29) we easily obtain the uniform estimate

$$\sup_{m \in \mathbb{N}} \mathbb{E} \int_0^T \|\sigma_m(s, \mathbf{u}^m(s), z)\|_{L^2(Z, \nu; H)}^2 ds \leq K_0 T + K_1 \mathbb{E} \int_0^T |\mathbf{u}^m(s)|^2 ds + K'_1 \mathbb{E} \int_0^T \|\mathbf{u}^m(s)\|_2 ds \leq C,$$

which implies that $\sigma_m(s, \mathbf{u}^m(s), z)$ is a bounded sequence in $L^2(\Omega \times [0, T]; L^2(Z, \nu; H))$. Therefore, by Banach-Alaoglu we deduce the existence of Σ belonging to $L^2(\Omega \times [0, T]; L^2(Z, \nu; H))$ such that (35) holds. This completes the proof of the proposition. \square

With the convergences in Proposition 3.5 we can pass to the limit in each term of (16) and obtain that

$$\mathbf{u}(t) + \kappa_1 \int_0^t \mathcal{A} \mathbf{u}(s) ds + \kappa_0 \int_0^t \mathbf{A}(s) ds = \mathbf{u}_0 + \int_0^t \int_Z \Sigma(s, z) \tilde{\eta}(dz, ds), \quad (36)$$

\mathbb{P} -a.s. and for any $t \in [0, T]$ as an equality in V^* . Also, passing to the limit in (29) gives the estimate in Theorem 3.2. Thanks to (33) and (35) we can deduce from [33] that the stochastic process \mathbf{u} has a càdlàg modification taking values in H . From now on we will identify \mathbf{u} with its càdlàg modification. Henceforth, we need to show the following identities to complete the proof of Theorem 3.2.

Proposition 3.6. *We have the following identities*

$$\mathbf{B} = B(\mathbf{u}, \mathbf{u}) \text{ in } L^2(\Omega \times [0, T]; V^*), \quad (37)$$

$$\mathbf{A} = \mathcal{A}_p \mathbf{u} \text{ in } L^2(\Omega \times [0, T]; V^*), \quad (38)$$

$$\Sigma = \sigma(t, \mathbf{u}, \cdot) \text{ in } L^2(\Omega \times [0, T]; L^2(Z, \nu; H)). \quad (39)$$

For any integer $M \geq 1$ we consider the sequence of stopping times $\{\tau_M : M \geq 1\}$ defined by

$$\tau_M = \inf\{t \in [0, T] : |\mathbf{u}(t)|^2 + \int_0^t \|\mathbf{u}(s)\|_2^2 ds \geq M^2\} \wedge T.$$

The proof of Proposition 3.6 requires the following convergences.

Lemma 3.7. *For any $M \geq 1$ we have that, as $m \rightarrow \infty$,*

$$1_{[0, \tau_M]}(\mathbf{u}^m - \mathbf{u}) \rightarrow 0 \text{ in } L^2(\Omega \times [0, T]; V), \quad (40)$$

and

$$\mathbb{E}|\mathbf{u}^m(\tau_M) - \mathbf{u}(\tau_M)| \rightarrow 0. \quad (41)$$

Proof of Lemma 3.7. Let $\tilde{\mathbf{u}}^m$ be the orthogonal projection of \mathbf{u} onto $\text{Span}\{\phi_1, \dots, \phi_m\}$, that is

$$\tilde{\mathbf{u}}^m = \sum_{i=1}^m (\mathbf{u}, \phi_i) \phi_i.$$

It is clear that as $m \rightarrow \infty$

$$\tilde{\mathbf{u}}^m \rightarrow \mathbf{u} \text{ in } L^2(\Omega \times [0, T]; H). \quad (42)$$

We also can check easily that

$$\mathbb{E}|\tilde{\mathbf{u}}^m(\tau_M) - \mathbf{u}(\tau_M)|^2 \rightarrow 0, \quad (43)$$

as $m \rightarrow \infty$.

First we should note that

$$\langle \mathcal{A}\tilde{\mathbf{u}}^m(t), \tilde{\mathbf{u}}^m(t) \rangle = \left\langle \sum_j \mathcal{A}\phi_j(\tilde{\mathbf{u}}^m(t), \phi_j), \sum_i (\tilde{\mathbf{u}}^m(t), \phi_i) \phi_i \right\rangle$$

Thanks to (15) we have

$$\begin{aligned} \langle \mathcal{A}\tilde{\mathbf{u}}^m(t), \tilde{\mathbf{u}}^m(t) \rangle &= \sum_{i,j} (\tilde{\mathbf{u}}^m(t), \phi_j) (\tilde{\mathbf{u}}^m(t), \phi_i) (\lambda_j \phi_j, \phi_i), \\ &= \sum_j (\tilde{\mathbf{u}}^m(t), \phi_j)^2 (\lambda_j \phi_j, \phi_j). \end{aligned}$$

Thanks to (15) again we have

$$\langle \mathcal{A}\tilde{\mathbf{u}}^m(t), \tilde{\mathbf{u}}^m(t) \rangle = \sum_j (\tilde{\mathbf{u}}^m(t), \phi_j)^2 \langle \mathcal{A}\phi_j, \phi_j \rangle.$$

From this, we can easily derive that

$$\|\tilde{\mathbf{u}}^m(t)\|_2^2 \leq \|\mathcal{A}\| |\mathbf{u}(t)|^2, \quad (44)$$

for almost all $(\omega, t) \in \Omega \times [0, T]$. Also,

$$\begin{aligned}
\|\tilde{\mathbf{u}}^m(s) - \mathbf{u}(s)\|_2^2 &\leq \langle \mathcal{A}\tilde{\mathbf{u}}^m(s) - \mathcal{A}\mathbf{u}(s), \tilde{\mathbf{u}}^m(s) - \mathbf{u}(s) \rangle \\
&\leq \langle \sum_{i=m+1}^{\infty} (\mathbf{u}(s), \mathcal{A}\phi_i)\phi_i, \sum_{j=m+1}^{\infty} (\mathbf{u}(s), \phi_j)\phi_j \rangle, \\
&\leq \langle \sum_{i=m+1}^{\infty} (\mathbf{u}(s), \phi_i)\tilde{\lambda}_i\phi_i, \sum_{j=m+1}^{\infty} (\mathbf{u}(s), \phi_j)\phi_j \rangle, \\
&\leq \sum_{j=m+1}^{\infty} (\mathbf{u}(s), \phi_j)^2 \langle \mathcal{A}\phi_j, \phi_j \rangle \\
&\leq \|\mathcal{A}\| \sum_{j=m+1}^{\infty} (\mathbf{u}(s), \phi_j)^2,
\end{aligned} \tag{45}$$

for any m . Since $\mathbf{u} \in H$ for almost all $(\omega, t) \in \Omega \times [0, T]$, we see that the right hand side of the last inequality converges to 0 as $m \rightarrow \infty$. Therefore

$$\tilde{\mathbf{u}}^m(s) \rightarrow \mathbf{u} \text{ in } V \text{ for almost all } (\omega, t) \in \Omega \times [0, T]. \tag{46}$$

Furthermore, owing to (44) and the dominated convergence theorem we can state that

$$\tilde{\mathbf{u}}^m \rightarrow \mathbf{u} \text{ in } L^2(\Omega \times [0, T]; V). \tag{47}$$

Next, it is not difficult to see that $\tilde{\mathbf{u}}^m$ satisfies the following equations

$$\tilde{\mathbf{u}}^m(t) + \kappa_1 \int_0^t \mathcal{A}\tilde{\mathbf{u}}^m(s)ds + \kappa_0 \int_0^t \Pi_m \mathbf{A}(s)ds + \int_0^t \Pi_m B(s) = \Pi_m \xi + \int_0^t \int_Z \Pi_m \Sigma(s, z) \tilde{\eta}(dz, ds).$$

Let X^m be the stochastic processes defined by $X^m = \mathbf{u}^m - \tilde{\mathbf{u}}^m$. From the equations of the last line and (16) we obtain

$$\begin{aligned}
X^m(t) + \kappa_0 \int_0^t [\mathcal{A}_p \mathbf{u}^m(s) - \Pi_m \mathbf{A}(s)]ds + \int_0^t [B_m(\mathbf{u}^m(s), \mathbf{u}^m(s)) - \Pi_m \mathbf{B}(s)]ds \\
= \int_0^t \int_Z [\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z)] \tilde{\eta}(dz, ds) - \kappa_1 \int_0^t \mathcal{A}X^m(s)ds.
\end{aligned}$$

Applying Itô's formula to the function $\Phi(x) = |x|^2$ and $X^m(t)$ yields

$$\begin{aligned}
|X^m(t)|^2 + 2\kappa_1 \int_0^t \langle \mathcal{A}X^m(s), X^m(s) \rangle ds + 2\kappa_0 \int_0^t \langle \mathcal{A}_p \mathbf{u}^m(s) - \Pi_m \mathbf{A}(s), X^m(s) \rangle ds \\
= 2 \int_0^t \langle \Pi_m \mathbf{B}(s) - B_m(\mathbf{u}^m(s), \mathbf{u}^m(s)), X^m(s) \rangle ds + \int_0^t \int_Z \Psi(s, z) \eta(dz, ds) \\
+ 2 \int_0^t \int_Z (\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z), X^m(s)) \tilde{\eta}(dz, ds),
\end{aligned}$$

where

$$\begin{aligned}
\Psi(s, z) &= |X^m(s-) + \sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z)|^2 - |X^m(s-)|^2 \\
&\quad - 2(\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z), X^m(s-)) \\
&= |\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z)|^2.
\end{aligned}$$

Let $r(t)$ be the real valued stochastic process defined by $r(t) = K_1 t + \frac{C_0^2}{4\kappa_1} \int_0^t \|\mathbf{u}(s)\|_2^2 ds$. Applying Itô's formula to $e^{-r(t)} |X^m(t)|^2$ leads to

$$\begin{aligned} & e^{-r(t)} |X^m(t)|^2 + 2\kappa_1 \int_0^t e^{-r(s)} \|X^m(s)\|_2^2 ds + 2\kappa_0 \int_0^t e^{-r(s)} \langle \mathcal{A}_p \mathbf{u}^m(s) - \Pi_m \mathbf{A}(s), X^m(s) \rangle ds \\ &= 2 \int_0^t e^{-r(s)} \langle \Pi_m \mathbf{B}(s) - B_m(\mathbf{u}^m(s), \mathbf{u}^m(s)), X^m(s) \rangle ds - \frac{C_0^2}{4\kappa_1} \int_0^t e^{-r(s)} |X^m(s)|^2 \|X^m(s)\|_2^2 ds \\ & \quad - K_1 \int_0^t e^{-r(s)} |X^m(s)|^2 ds + \int_0^t \int_Z e^{-r(s)} |\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z)|^2 \eta(dz, ds) \\ & \quad + 2 \int_0^t e^{-r(s)} \int_Z (\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z), X^m(s)) \tilde{\eta}(dz, ds), \end{aligned} \quad (48)$$

Let us study each term of (48). For the nonlinear term involving B_m and \mathbf{B} we have that

$$B_m(\tilde{\mathbf{u}}^m, \tilde{\mathbf{u}}^m) - B_m(\mathbf{u}^m, \mathbf{u}^m) = B_m(X^m, \tilde{\mathbf{u}}^m) + B_m(\mathbf{u}^m, X^m). \quad (49)$$

Out of this and (8) we obtain that

$$\begin{aligned} \langle \Pi_m \mathbf{B}(s) - B_m(\mathbf{u}^m(s), \mathbf{u}^m(s)), X^m(s) \rangle &= \langle \Pi_m \mathbf{B}(s) - B_m(\tilde{\mathbf{u}}^m(s), \tilde{\mathbf{u}}^m(s)), X^m(s) \rangle \\ & \quad + \langle B_m(X^m(s), \tilde{\mathbf{u}}^m(s)), X^m(s) \rangle, \end{aligned}$$

which along with (9) and Young's inequality imply that

$$\begin{aligned} \langle \Pi_m \mathbf{B}(s) - B_m(\mathbf{u}^m(s), \mathbf{u}^m(s)), X^m(s) \rangle &\leq \langle \Pi_m \mathbf{B}(s) - B_m(\tilde{\mathbf{u}}^m(s), \tilde{\mathbf{u}}^m(s)), X^m(s) \rangle \\ & \quad + \frac{C_0^2}{4\kappa_1} |X^m(s)|^2 \|\tilde{\mathbf{u}}^m(s)\|_2^2 + \kappa_1 \|X^m(s)\|_2^2. \end{aligned} \quad (50)$$

Next, we have

$$\begin{aligned} \langle \mathcal{A}_p \mathbf{u}^m(s) - \mathbf{A}(s), X^m(s) \rangle &= \langle \mathcal{A}_p \mathbf{u}^m(s) - \mathcal{A}_p \tilde{\mathbf{u}}^m(s), X^m(s) \rangle \\ & \quad + \langle \mathcal{A}_p \tilde{\mathbf{u}}^m(s) - \mathbf{A}(s), X^m(s) \rangle. \end{aligned} \quad (51)$$

Invoking the point (ii) of Lemma 2.3 we see that

$$\langle \mathcal{A}_p \mathbf{u}^m(s) - \mathcal{A}_p \tilde{\mathbf{u}}^m(s), X^m(s) \rangle \geq 0. \quad (52)$$

Setting $S = |\sigma_m(s, \mathbf{u}^m(s), z) - \Pi_m \Sigma(s, z)|^2$ we see that

$$\begin{aligned} S &= |\Pi_m[\sigma(s, \mathbf{u}^m(s), z) - \sigma(s, \mathbf{u}(s), z)]|^2 - |\Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s, z)]|^2 \\ & \quad + 2(\Pi_m[\sigma(s, \mathbf{u}^m(s), z) - \Sigma(s, z)], \Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s, z)]). \end{aligned}$$

Owing to point (1) of Condition 1 we have that

$$\begin{aligned} S &\leq \ell_2 |X^m(s)|^2 + \ell_2 \|\tilde{\mathbf{u}}^m(s) - \mathbf{u}(s)\|^2 - |\Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s, z)]|^2 \\ & \quad + 2(\Pi_m[\sigma(s, \mathbf{u}^m(s), z) - \Sigma(s, z)], \Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s, z)]). \end{aligned} \quad (53)$$

Putting (50), (51), (52) and (53) into (48), replacing t by τ_M and taking the mathematical expectation lead to

$$\begin{aligned}
& \mathbb{E} e^{-r(\tau_M)} |X^m(\tau_M)|^2 + \mathbb{E} \int_0^{\tau_M} \int_Z e^{-r(s)} |\Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s, z)]|^2 \eta(dz, ds) \\
& \leq -\kappa_1 \mathbb{E} \int_0^{\tau_M} e^{-r(s)} \|X^m(s)\|_2^2 ds + 2\kappa_0 \mathbb{E} \int_0^{\tau_M} e^{-r(s)} \langle \Pi_m \mathbf{A}(s) - \mathcal{A}_p \tilde{\mathbf{u}}^m(s), X^m(s) \rangle ds \\
& + 2\mathbb{E} \int_0^{\tau_M} \int_Z e^{-r(s)} (\Pi_m[\sigma(s, \mathbf{u}^m(s), z) - \Sigma(s, z)], \Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s, z)]) \eta(dz, ds) \\
& + \mathbb{E} \int_0^{\tau_M} e^{-r(s)} \langle \Pi_m \mathbf{B}(s) - B_m(\tilde{\mathbf{u}}^m(s), \tilde{\mathbf{u}}^m(s)), X^m(s) \rangle ds \\
& + K_1 \mathbb{E} \int_0^{\tau_M} |\tilde{\mathbf{u}}^m(s) - \mathbf{u}(s)|^2 e^{-r(s)} ds.
\end{aligned} \tag{54}$$

Now we will show that the last four terms of the right hand side of (54) will tend to 0 as $m \rightarrow 0$. Thanks to (42) we have

$$\mathbb{E} \int_0^T 1_{[0, \tau_M]}(s) e^{-r(s)} |\tilde{\mathbf{u}}^m(s) - \mathbf{u}(s)|^2 ds \rightarrow 0. \tag{55}$$

Owing to (49) and (9) we see that

$$\begin{aligned}
& \left\| 1_{[0, \tau_M]}(t) e^{-r(t)} [B(\tilde{\mathbf{u}}^m(t), \tilde{\mathbf{u}}^m(t)) - B(\mathbf{u}(t), \mathbf{u}(t))] \right\|_{V^*} \leq 1_{[0, \tau_M]}(t) C_0 \|\tilde{\mathbf{u}}^m(t)\|_1 |\tilde{\mathbf{u}}^m(t) - \mathbf{u}(t)| \\
& + 1_{[0, \tau_M]}(t) C_0 |\mathbf{u}(t)| \|\tilde{\mathbf{u}}^m(t) - \mathbf{u}(t)\|_2,
\end{aligned} \tag{56}$$

which with (46) implies that

$$\left\| 1_{[0, \tau_M]}(t) e^{-r(t)} [B(\tilde{\mathbf{u}}^m(t), \tilde{\mathbf{u}}^m(t)) - B(\mathbf{u}(t), \mathbf{u}(t))] \right\|_{V^*} \rightarrow 0 \text{ a.e. } (\omega, t) \in \Omega \times [0, T],$$

as $m \rightarrow \infty$. Furthermore, owing to (44) and (45) we see from (56) that

$$\left\| 1_{[0, \tau_M]}(t) e^{-r(t)} [B(\tilde{\mathbf{u}}^m(t), \tilde{\mathbf{u}}^m(t)) - B(\mathbf{u}(t), \mathbf{u}(t))] \right\|_{V^*} \leq 2C_0 M \|A\|^{\frac{1}{2}} |\mathbf{u}(t)|. \tag{57}$$

Note that $|\mathbf{u}(t)|$ is bounded in $L^2(\Omega \times [0, T], \mathbb{R})$. Thus, the Dominated Convergence Theorem implies that

$$\left\| 1_{[0, \tau_M]}(t) e^{-r(t)} [B(\tilde{\mathbf{u}}^m(t), \tilde{\mathbf{u}}^m(t)) - B(\mathbf{u}(t), \mathbf{u}(t))] \right\|_{V^*} \rightarrow 0 \text{ in } L^2(\Omega \times [0, T]; \mathbb{R}) \tag{58}$$

By the convergences (32) and (47) we have

$$\tilde{\mathbf{u}}^m - \mathbf{u}^m \rightharpoonup 0 \text{ in } L^2(\Omega; L^2(0, T; V)). \tag{59}$$

We derive from this, (57) and (58) that

$$E \int_0^{\tau_M} e^{-r(s)} \langle B(\tilde{\mathbf{u}}^m(s), \tilde{\mathbf{u}}^m(s)) - B(\mathbf{u}(s), \mathbf{u}(s)), \tilde{\mathbf{u}}^m(s) - \mathbf{u}^m(s) \rangle ds \rightarrow 0$$

as $m \rightarrow \infty$. Hence

$$\begin{aligned}
& \lim_{m \rightarrow \infty} E \int_0^{\tau_M} e^{-r(s)} \langle B(\tilde{\mathbf{u}}^m(s), \tilde{\mathbf{u}}^m(s)) - B^*(s), \tilde{\mathbf{u}}^m(s) - \mathbf{u}^m(s) \rangle ds \\
& = \lim_{m \rightarrow \infty} E \int_0^{\tau_M} e^{-r(s)} \langle B(\tilde{\mathbf{u}}^m(s), \tilde{\mathbf{u}}^m(s)) - B(\mathbf{u}(s), \mathbf{u}(s)), \tilde{\mathbf{u}}^m(s) - \mathbf{u}^m(s) \rangle ds \\
& + \lim_{m \rightarrow \infty} E \int_0^{\tau_M} e^{-r(s)} \langle B(\mathbf{u}, \mathbf{u}) - B^*(s), \tilde{\mathbf{u}}^m(s) - \mathbf{u}^m(s) \rangle ds \\
& = 0.
\end{aligned}$$

Since $\Pi_m \circ \Pi_m = \Pi_m$ and $\|\Pi_m\| \leq 1$, it follows that $1_{[0, \tau_M]} e^{-r(s)} \Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s, z)]$ is bounded in $L^2(\Omega \times [0, T]; L^2(Z, \nu; H))$. Therefore we see from (35) that

$$2\mathbb{E} \int_0^{\tau_M} \int_Z e^{-r(s)} (\Pi_m[\sigma(s, \mathbf{u}^m(s), z) - \Sigma(s, z)], \Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s, z)]) \eta(dz, ds) \rightarrow 0$$

as $m \rightarrow \infty$.

Now it is not difficult to check that

$$\begin{aligned} \mathbb{E} \int_0^t e^{-r(s)} \langle \Pi_m[\mathbf{A}(s) - \mathcal{A}_p \tilde{\mathbf{u}}^m(s)], X^m(s) \rangle ds &= \mathbb{E} \int_0^t e^{-r(s)} \langle \Pi_m[\mathbf{A}(s) - \mathcal{A}_p \mathbf{u}(s)], X^m(s) \rangle ds \\ &\quad + \mathbb{E} \int_0^t e^{-r(s)} \langle \Pi_m[\mathcal{A}_p \mathbf{u}(s) - \mathcal{A} \tilde{\mathbf{u}}^m(s)], X^m(s) \rangle ds. \end{aligned}$$

Since $\langle \Pi_m \mathbf{v}, X^m \rangle = \langle \mathbf{v}, X^m \rangle$ for any $\mathbf{v} \in V^*$ and $e^{-r(s)}(\mathbf{A}(s) - \mathcal{A}_p \mathbf{u}(s))$ is a bounded element of $L^2(\Omega \times [0, T]; V^*)$, we derive from (59) that the first term of the right hand side of the above equation tends to zero as $m \rightarrow \infty$. Owing to point (i) of Lemma 2.3, the strong convergence (47) and the weak convergence (59) we see that the second term of the right hand side converges to zero as well. Thus, we have just proved that

$$\mathbb{E} \int_0^t e^{-r(s)} \langle \Pi_m \mathbf{A}(s) - \mathcal{A} \tilde{\mathbf{u}}^m(s), X^m(s) \rangle ds \rightarrow 0,$$

as $m \rightarrow \infty$. With this we have just shown that the last four terms of (54) converges to zero as $m \rightarrow \infty$. Then, we can conclude with that

$$\mathbb{E} e^{-r(\tau_M)} |X^m(\tau_M)|^2 + \kappa_1 \mathbb{E} \int_0^{\tau_M} e^{-r(s)} \|X^m(s)\|_2^2 ds \rightarrow 0, \quad (60)$$

$$\mathbb{E} \int_0^{\tau_M} \int_Z e^{-r(s)} |\Pi_m[\sigma(s, \mathbf{u}(s), z) - \Sigma(s, z)]|^2 \eta(dz, ds) \rightarrow 0, \quad (61)$$

as $m \rightarrow \infty$. We easily terminate the proof of the lemma by putting equations (43) and (47) into (60). \square

Proof of Proposition 3.6. First note that for any $\mathbf{w} \in V$

$$\begin{aligned} S &= \langle B(\mathbf{u}^m, \mathbf{u}^m) - B(\mathbf{u}, \mathbf{u}), \mathbf{w} \rangle \\ &= \langle B(\mathbf{u}^m - \mathbf{u}, \mathbf{u}^m), \mathbf{w} \rangle + \langle B(\mathbf{u}, \mathbf{u}^m - \mathbf{u}), \mathbf{w} \rangle. \end{aligned} \quad (62)$$

The following equations also hold true

$$\begin{aligned} \langle B(\mathbf{u}^m - \mathbf{u}, \mathbf{u}^m), \mathbf{w} \rangle &= \langle B(\mathbf{u}^m, \mathbf{u}^m), \mathbf{w} \rangle - \langle B(\mathbf{u}, \mathbf{u}^m), \mathbf{w} \rangle, \\ \langle B(\mathbf{u}^m, \mathbf{u} - \mathbf{u}^m), \mathbf{w} \rangle &= \langle B(\mathbf{u}^m, \mathbf{u}), \mathbf{w} \rangle - \langle B(\mathbf{u}^m, \mathbf{u}^m), \mathbf{w} \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} S &= \langle B(\mathbf{u}, \mathbf{u} - \mathbf{u}^m), \mathbf{w} \rangle - \langle B(\mathbf{u}^m, \mathbf{u} - \mathbf{u}^m), \mathbf{w} \rangle + \langle B(\mathbf{u}^m, \mathbf{u}), \mathbf{w} \rangle \\ &\quad - \langle B(\mathbf{u}, \mathbf{u}^m), \mathbf{w} \rangle. \end{aligned} \quad (63)$$

The operator

$$\begin{aligned} B_{\mathbf{a}, \cdot} : V &\rightarrow V^* \\ \mathbf{v} &\mapsto B_{\mathbf{a}, \cdot}(\mathbf{v}) = B(\mathbf{a}, \mathbf{v}) \end{aligned}$$

is linear continuous for any fixed $\mathbf{a} \in V$. Due to this fact and (32), it is true that

$$B(\mathbf{u}, \mathbf{u}^m) \rightharpoonup B(\mathbf{u}, \mathbf{u}) \text{ weakly in } L^2(\Omega \times [0, T]; V^*). \quad (64)$$

By a similar argument, we also prove the following convergence

$$B(\mathbf{u}^m, \mathbf{u}) \rightharpoonup B(\mathbf{u}, \mathbf{u}) \text{ weakly in } L^2(\Omega \times [0, T]; V^*). \quad (65)$$

Now let \mathbf{w} be an element of $L^\infty(\Omega \times [0, T]; V)$. We deduce from the property (9) that

$$\begin{aligned} & \left| \mathbb{E} \int_0^T 1_{[0, \tau_M]} \langle B(\mathbf{u}(s), \mathbf{u}(s) - \mathbf{u}^m(s)), \mathbf{w}(s) \rangle - \langle B(\mathbf{u}^m(s), \mathbf{u}(s) - \mathbf{u}^m(s)), \mathbf{w}(s) \rangle ds \right| \\ & \leq C \mathbb{E} \int_0^{\tau_M} |\mathbf{u}(s)| \|\mathbf{u}^m(s) - \mathbf{u}(s)\|_2 ds + C \mathbb{E} \int_0^{\tau_M} |\mathbf{u}^m(s)| \|\mathbf{u}^m(s) - \mathbf{u}(s)\|_2 ds, \end{aligned}$$

from which and (40) we derive that

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^T 1_{[0, \tau_M]} \langle B(\mathbf{u}(s), \mathbf{u}(s) - \mathbf{u}^m(s)), \mathbf{w}(s) \rangle - \langle B(\mathbf{u}^m(s), \mathbf{u}(s) - \mathbf{u}^m(s)), \mathbf{w}(s) \rangle ds = 0 \quad (66)$$

Since $\tau_M \nearrow T$ almost surely and $L^\infty(\Omega \times [0, T]; V)$ is dense in $L^2(\Omega \times [0, T]; V)$, we deduce from (63)-(66) that the identity (37) holds.

Next, thanks to the property of \mathcal{A}_p (mainly (10)) we see that

$$\mathbb{E} \int_0^{\tau_M} \|\mathcal{A}_p(\mathbf{u}^m(s)) - \mathcal{A}_p(\mathbf{u})\|_{V^*}^2 ds \leq C \mathbb{E} \int_0^{\tau_M} \|\mathbf{u}^m - \mathbf{u}\|_2^2 ds.$$

Owing to (40) and the fact that $\tau_M \nearrow T$ almost surely as $M \rightarrow \infty$, we obtain the equation (38).

The identity (39) easily follows from (61). This completes the proof of the Proposition 3.6. \square

4. PATHWISE UNIQUENESS AND CONVERGENCE OF THE WHOLE SEQUENCE OF GALERKIN APPROXIMATION

In this section we show the pathwise uniqueness of the solution and some (strong) convergences of the Galerkin approximate solution to the exact solution of (1).

Theorem 4.1. *Let \mathbf{u}_1 and \mathbf{u}_2 be two strong solutions to (6) defined on the same stochastic system $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \tilde{\eta})$. Let ξ_1 and ξ_2 be their respective initial conditions. Then for any $t \in [0, T]$ we have*

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2 \leq C(\omega) |\xi_1 - \xi_2|^2,$$

almost surely.

Proof. Let \mathbf{u}_1 (resp., \mathbf{u}_2) be a strong solution to (6) with initial condition ξ_1 (resp., ξ_2). Let $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ and $\xi = \xi_1 - \xi_2$. It is not hard to see that

$$\begin{aligned} & \mathbf{w}(t) + \kappa_1 \int_0^t A \mathbf{w}(s) ds + \kappa_0 \int_0^t (\mathcal{A}_p \mathbf{u}_1(s) - \mathcal{A}_p \mathbf{u}_2(s)) ds \\ & = \xi + \int_0^t \int_Z (\sigma(s, \mathbf{u}_1(s), z) - \sigma(s, \mathbf{u}_2(s), z)) \tilde{\eta}(dz, ds) \\ & \quad + \int_0^t (B(\mathbf{u}_2(s), \mathbf{u}_2(s)) - B(\mathbf{u}_1(s), \mathbf{u}_1(s))) ds. \end{aligned}$$

Applying Itô's formula to the function $\Phi(x) = |x|^2$ and $\mathbf{w}(t)$ implies that

$$\begin{aligned} & |\mathbf{w}(t)|^2 + 2\kappa_1 \int_0^t \|\mathbf{w}(s)\|_2^2 ds + 2\kappa_0 \int_0^t \langle \mathcal{A}_p \mathbf{u}_1(s) - \mathcal{A}_p \mathbf{u}_2(s), \mathbf{w}(s) \rangle ds \\ & = |\xi|^2 + 2 \int_0^t \int_Z (\sigma(s, \mathbf{u}_1(s), z) - \sigma(s, \mathbf{u}_2(s), z), \mathbf{w}(s)) \tilde{\eta}(dz, ds) \\ & \quad - 2 \int_0^t \langle B(\mathbf{u}_1(s), \mathbf{u}_1(s)) - B(\mathbf{u}_2(s), \mathbf{u}_2(s)), \mathbf{w}(s) \rangle ds \\ & \quad + \int_0^t \int_Z |\sigma(s, \mathbf{u}_1(s), z) - \sigma(s, \mathbf{u}_2(s), z)|^2 \eta(dz, ds). \end{aligned}$$

Next we introduce the real valued process

$$\rho(t) = e^{-\frac{C_0^2}{\kappa_1} \int_0^t \|\mathbf{u}_1(s)\|_2^2 ds}.$$

Now we apply Itô's formula to $\rho(t)|\mathbf{w}(t)|^2$ and we get

$$\begin{aligned} & \rho(t)|\mathbf{w}(t)|^2 + 2\kappa_1 \int_0^t \rho(s) \|\mathbf{w}(s)\|_2^2 ds + 2\kappa_0 \int_0^t \rho(s) \langle \mathcal{A}_p \mathbf{u}_1(s) - \mathcal{A}_p \mathbf{u}_2(s), \mathbf{w}(s) \rangle ds \\ &= -2 \int_0^t \rho(s) \langle B(\mathbf{u}_1(s), \mathbf{u}_1(s)) - B(\mathbf{u}_2(s), \mathbf{u}_2(s)), \mathbf{w}(s) \rangle ds - \frac{C_0^2}{\kappa_1} \int_0^t \rho(s) |\mathbf{w}(s)|^2 \|\mathbf{u}_1(s)\|_2^2 ds \\ & \quad + 2 \int_0^t \int_Z \rho(s) (\sigma(s, \mathbf{u}_1(s), z) - \sigma(s, \mathbf{u}_2(s), z), \mathbf{w}(s)) \tilde{\eta}(dz, ds) \\ & \quad + \int_0^t \int_Z \rho(s) |\sigma(s, \mathbf{u}_1(s), z) - \sigma(s, \mathbf{u}_2(s), z)|^2 \eta(dz, ds) + |\xi|^2. \end{aligned}$$

By making use of (11), (62), (8), (9) and Young's inequality with $\varepsilon = \kappa_1$ in the above estimate and by taking the mathematical expectation to both sides of the resulting estimate yield

$$\mathbb{E} \rho(t) |\mathbf{w}(t)|^2 + 2\kappa_1 \mathbb{E} \int_0^t \rho(s) \|\mathbf{w}(s)\|_2^2 ds \leq \mathbb{E} \int_0^t \int_Z \rho(s) |\sigma(s, \mathbf{u}_1(s), z) - \sigma(s, \mathbf{u}_2(s), z)|^2 \nu(dz) ds + |\xi|^2.$$

Using point (1) of Condition 1 yields that

$$\mathbb{E} \rho(t) |\mathbf{w}(t)|^2 \leq \mathbb{E} |\xi|^2 + \mathbb{E} \int_0^t \rho(s) |\mathbf{w}(s)|^2 ds,$$

from which and Gronwall's Lemma we deduce the existence of a constant $C > 0$ such that

$$\mathbb{E} \rho(t) |\mathbf{w}(t)|^2 \leq C \mathbb{E} |\xi|^2,$$

for any $t \in [0, T]$. Since $\rho(t)$ is bounded \mathbb{P} -a.s, we conclude easily from the last estimate the proof of the theorem. \square

Next we will show that the whole sequence of solutions to the Galerkin approximation system (16) converges in mean square to the exact strong solution of (1). Mainly we have

Theorem 4.2. *The whole sequence of Galerkin approximation $\{\mathbf{u}^m : m \in \mathbb{N}\}$ defined by (16) satisfies*

$$\lim_{m \rightarrow \infty} \mathbb{E} |\mathbf{u}^m(T-) - \mathbf{u}(T-)|^2 = 0, \quad (67)$$

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{T-} \|\mathbf{u}^m(s) - \mathbf{u}(s)\|_2^2 ds = 0. \quad (68)$$

The main ingredient of the proof of this result is the following lemma, its proof follows a very small modification of the proof of [11, Proposition B.3].

Lemma 4.3. *Let $\{Q_m; m \geq 1\} \subset L^2(\Omega \times [0, T]; \mathbb{R})$ be a sequence of càdlàg real-valued process, and let $\{T_M; M \geq 1\}$ be a sequence of \mathcal{F}^t -stopping times such that T_M is increasing to T , $\sup_{m \geq 1} \mathbb{E} |Q_m(T)|^2 < \infty$, and $\lim_{m \rightarrow \infty} \mathbb{E} |Q_m(T_M)| = 0$ for all $M \geq 1$. Then $\lim_{m \rightarrow \infty} \mathbb{E} |Q_m(T-)| = 0$.*

Proof of Theorem 4.2. It follows from Lemma 3.7 that

$$\lim_{m \rightarrow \infty} \mathbb{E} \int_0^{\tau_M} \|\mathbf{u}^m(t) - \mathbf{u}(t)\|_2^2 dt = 0, \quad (69)$$

and

$$\lim_{m \rightarrow \infty} \mathbb{E} |\mathbf{u}^m(\tau_M) - \mathbf{u}(\tau_M)|^2 = 0, \quad (70)$$

for any $M \geq 1$. So by applying the preceding lemma to $Q_m(t) = \|\mathbf{u}^m(t) - \mathbf{u}(t)\|^2$, $T_M = \tau_M$ and taking into account (70), the estimates in Proposition 3.4 and the uniqueness of \mathbf{u} , we see that

the whole sequence \mathbf{u}^m defined by (16) satisfies (67). To prove (68) we need an extra estimate for the sequence $\{\mathbf{u}^m : m \in \mathbb{N}\}$. Since

$$\int_0^t \int_Z \Psi(s, z) \tilde{\eta}(dz, ds) = \int_0^t \int_Z \Psi(s, z) \eta(ds, dz) - \int_0^t \int_Z \Psi(s, z) \nu(dz) ds,$$

and $|\mathbf{v}|^2 + 2(\mathbf{v}, \mathbf{w}) = |\mathbf{v} + \mathbf{w}|^2 - |\mathbf{w}|^2$ we deduce from (18) that

$$\begin{aligned} 4\kappa_1^2 \mathbb{E} \left(\int_0^t \|\mathbf{u}^m(s)\|_2^2 ds \right)^2 &\leq 2\mathbb{E}|\xi|^4 + 4\mathbb{E} \left(\int_0^t \int_Z |\sigma(s, \mathbf{u}^m(s), z)| \nu(dz) ds \right)^2 \\ &\quad + 4\mathbb{E} \left(\int_0^t \int_Z \left[|\sigma(s, \mathbf{u}^m(s), z) + \mathbf{u}^m(s-)|^2 - |\mathbf{u}^m(s-)|^2 \right] \tilde{\eta}(dz, ds) \right)^2. \end{aligned}$$

Using item (1) of Condition 1 and the estimate in Proposition 3.4 we infer from the last inequality that

$$\begin{aligned} 4\kappa_1^2 \mathbb{E} \left(\int_0^t \|\mathbf{u}^m(s)\|_2^2 ds \right)^2 &\leq 4\mathbb{E} \left(\int_0^t \int_Z \left[|\sigma(s, \mathbf{u}^m(s), z) + \mathbf{u}^m(s-)|^2 - |\mathbf{u}^m(s-)|^2 \right] \tilde{\eta}(dz, ds) \right)^2 \\ &\quad + 2\mathbb{E}|\xi|^4 + 4\ell_0^2 T^2 + 4C\ell_1^2 T^2 (\mathbb{E}|\xi|^4 + 1). \end{aligned}$$

Now invoking [51, Theorem 4.14] we see that

$$\begin{aligned} 4\kappa_1^2 \mathbb{E} \left(\int_0^t \|\mathbf{u}^m(s)\|_2^2 ds \right)^2 &\leq 4\mathbb{E} \int_0^t \int_Z \left[\left| |\sigma(s, \mathbf{u}^m(s), z) + \mathbf{u}^m(s-)|^2 - |\mathbf{u}^m(s-)|^2 \right|^2 \right] \nu(dz) ds \\ &\quad + 2\mathbb{E}|\xi|^4 + 4\ell_0^2 T^2 + 4C\ell_1^2 T^2 (\mathbb{E}|\xi|^4 + 1), \end{aligned}$$

from which with item (3) of Condition 1 and Proposition 3.4 we derive that

$$4\kappa_1^2 \mathbb{E} \left(\int_0^t \|\mathbf{u}^m(s)\|_2^2 ds \right)^2 \leq (2 + 4C\ell_1^2 T^2 + C\ell_3 T + CT) \mathbb{E}|\xi|^4 + 4\ell_0^2 T^2 + 4C\ell_1^2 T^2 + CT.$$

This also implies that

$$4\kappa_1^2 \mathbb{E} \left(\int_0^t \|\mathbf{u}(s)\|_2^2 ds \right)^2 \leq (2 + 4C\ell_1^2 T^2 + C\ell_3 T + CT) \mathbb{E}|\xi|^4 + 4\ell_0^2 T^2 + 4C\ell_1^2 T^2 + CT.$$

We see easily from the last two estimates and (69) that $Q_m(t) = \int_0^t \|\mathbf{u}^m(s) - \mathbf{u}(s)\|_2^2 ds$, $T_M = \tau_M$ satisfy the hypotheses of the above lemma, therefore we can deduce that (68) holds. This ends the proof of Theorem 4.2. \square

5. EXISTENCE AND ERGODICITY OF INVARIANT MEASURE

In this section we are interested in the study of some qualitative properties of the solution of (1). We will mainly analyse the Markov, Fellerian properties of the solution. We will also derive the existence of ergodic invariant measures. To start with our investigation we denote by $\mathbf{u}(t; \xi)$ the solution of (1) with initial condition $\xi \in H$, and by $C_b(H)$ we describe the space of all continuous real-valued functionals defined on H . Next, we define the family of mappings $\{\mathcal{P}_t, t \geq 0\}$ (\mathcal{P}_t for short) defined on $C_b(H)$ by

$$\mathcal{P}_t \phi(\xi) = \mathbb{E} \phi(\mathbf{u}(t; \xi)),$$

for any $\phi \in C_b(H)$, $\xi \in H$, and $t \geq 0$. Thanks to Theorem 3.2 and Theorem 4.1 the family \mathcal{P}_t defines a semigroup on $C_b(H)$. More properties of the solution $\mathbf{u}(t; \xi)$ and the semigroup \mathcal{P}_t are given in the following results.

Theorem 5.1. *The solution $\mathbf{u}(t; \xi)$ defines a Markov process and the semigroup \mathcal{P}_t is Fellerian; that is, \mathcal{P}_t satisfies the following*

$$\mathcal{P}_t(C_b(H)) \subset C_b(H),$$

for any $t \geq 0$.

Before we proceed to the proof of these statements let us give an auxillary result.

Lemma 5.2. *Let $\mathbf{u}(t; \xi_1)$ and $\mathbf{u}(t; \xi_2)$ be two solutions of (1) associated to two distincts initial conditions ξ_1, ξ_2 , let*

$$\tau_R^\xi = \inf\{t : |\mathbf{u}(t; \xi)| > R\}, \forall R > 0, \xi \in H. \quad (71)$$

Let us set $\tau_R^{\xi_1, \xi_2} = \tau_R^{\xi_1} \wedge \tau_R^{\xi_2}$, $t_R = t \wedge \tau_R^{\xi_1, \xi_2}$ and $\mathbf{w}(t) = \mathbf{u}(t; \xi_1) - \mathbf{u}(t; \xi_2)$, $t \in [0, \infty)$. Then for any $R > 0$ and $t \in [0, \infty)$ there exists a positive constant C such that

$$\mathbb{E}|\mathbf{w}(t_R)|^2 \leq C\mathbb{E}|\xi_1 - \xi_2|^2. \quad (72)$$

Proof of Lemma 5.2. As in the proof of Theorem 4.1 we can check by making use of Itô's formula that $|\mathbf{w}(t_R)|^2$ satisfies

$$\begin{aligned} |\mathbf{w}(t_R)|^2 + 2\kappa_1 \int_0^{t_R} \|\mathbf{u}(s; \xi)\|_2^2 ds &\leq |\xi_1 - \xi_2|^2 + 2 \int_0^{t_R} \langle B(\mathbf{w}(s), \mathbf{u}(s; \xi_1), \mathbf{w}(s)) \rangle ds \\ &\quad + \int_0^{t_R} \int_Z |\sigma(s, \mathbf{u}(s; \xi_1), z) - \sigma(s, \mathbf{u}(s; \xi_2), z)|^2 \eta(dz, ds) \\ &\quad + 2 \int_0^{t_R} \int_Z (\sigma(s, \mathbf{u}(s; \xi_1), z) - \sigma(s, \mathbf{u}(s; \xi_2), z), \mathbf{w}(s-)) \tilde{\eta}(dz, ds). \end{aligned}$$

Using the skew-symmetry of B and Hölder's inequality we derive from the last inequality that

$$\begin{aligned} |\mathbf{w}(t_R)|^2 + 2\kappa_1 \int_0^{t_R} \|\mathbf{u}(s; \xi)\|_2^2 ds &\leq |\xi_1 - \xi_2|^2 + 2C \int_0^{t_R} \left(|\mathbf{w}(s) \cdot \nabla \mathbf{w}(s)| \times |\mathbf{u}(s; \xi_1)| \right) ds \\ &\quad + \int_0^{t_R} \int_Z |\sigma(s, \mathbf{u}(s; \xi_1), z) - \sigma(s, \mathbf{u}(s; \xi_2), z)|^2 \eta(dz, ds) \\ &\quad + 2 \int_0^{t_R} \int_Z (\sigma(s, \mathbf{u}(s; \xi_1), z) - \sigma(s, \mathbf{u}(s; \xi_2), z), \mathbf{w}(s-)) \tilde{\eta}(dz, ds). \end{aligned}$$

Owing to Hölder's inequality and the fact $|\mathbf{u}(s; \xi_1)| \geq R$ on $[0, t_R]$ we infer the existence of a constant $C_R = C(R) > 0$ such that

$$\begin{aligned} |\mathbf{w}(t_R)|^2 + 2\kappa_1 \int_0^{t_R} \|\mathbf{u}(s; \xi)\|_2^2 ds &\leq |\xi_1 - \xi_2|^2 + 2C_R \int_0^{t_R} \left(|\mathbf{w}(s)| \times |\nabla \mathbf{w}(s)|_{L^q} \right) ds \\ &\quad + \int_0^{t_R} \int_Z |\sigma(s, \mathbf{u}(s; \xi_1), z) - \sigma(s, \mathbf{u}(s; \xi_2), z)|^2 \eta(dz, ds) \\ &\quad + 2 \int_0^{t_R} \int_Z (\sigma(s, \mathbf{u}(s; \xi_1), z) - \sigma(s, \mathbf{u}(s; \xi_2), z), \mathbf{w}(s-)) \tilde{\eta}(dz, ds), \end{aligned}$$

where $2 < q \leq \frac{2n}{n-2}$. Thanks to Young's inequality and the continuous embedding $\mathcal{H}^1 \subset L^q$ we easily see that

$$\begin{aligned} |\mathbf{w}(t_R)|^2 + 2\kappa_1 \int_0^{t_R} \|\mathbf{u}(s; \xi)\|_2^2 ds &\leq |\xi_1 - \xi_2|^2 + \frac{2C_R}{\varepsilon} \int_0^{t_R} |\mathbf{w}(s)|^2 ds + \varepsilon \int_0^{t_R} \|\mathbf{w}(s)\|_2^2 ds \\ &\quad + \int_0^{t_R} \int_Z |\sigma(s, \mathbf{u}(s; \xi_1), z) - \sigma(s, \mathbf{u}(s; \xi_2), z)|^2 \eta(dz, ds) \\ &\quad + 2 \int_0^{t_R} \int_Z (\sigma(s, \mathbf{u}(s; \xi_1), z) - \sigma(s, \mathbf{u}(s; \xi_2), z), \mathbf{w}(s-)) \tilde{\eta}(dz, ds), \end{aligned}$$

Choosing $\varepsilon = \kappa_1$, using item (2) of Condition 1 and taking the mathematical expectation yield that

$$\mathbb{E}|\mathbf{w}(t_R)|^2 + \kappa_1 \mathbb{E} \int_0^{t_R} \|\mathbf{u}(s; \xi)\|_2^2 ds \leq |\xi_1 - \xi_2|^2 + \left(\frac{2C_R}{\kappa_1} + L_1 \right) \int_0^{t_R} \mathbb{E}|\mathbf{w}(s)|^2 ds. \quad (73)$$

where we have used the fact that

$$\begin{aligned} & \mathbb{E} \int_0^{t_R} \int_Z |\sigma(s, \mathbf{u}(s; \xi_1), z) - \sigma(s, \mathbf{u}(s; \xi_2), z)|^2 \eta(dz, ds) \\ &= \mathbb{E} \int_0^{t_R} |\sigma(s, \mathbf{u}(s; \xi_1), z) - \sigma(s, \mathbf{u}(s; \xi_2), z)|^2 \nu(dz) ds, \end{aligned}$$

and

$$2\mathbb{E} \int_0^{t_R} \int_Z (\sigma(s, \mathbf{u}(s; \xi_1), z) - \sigma(s, \mathbf{u}(s; \xi_2), z), \mathbf{w}(s-)) \tilde{\eta}(dz, ds) = 0.$$

Notice that (73) can be rewritten in the following form

$$\mathbb{E} |\mathbf{w}(t_R)|^2 + \kappa_1 \mathbb{E} \int_0^{t_R} \|\mathbf{u}(s; \xi)\|_2^2 ds \leq |\xi_1 - \xi_2|^2 + \left(\frac{2C_R}{\kappa_1} + L_1 \right) \int_0^t |\mathbf{w}(s \wedge \tau_R)|^2 ds,$$

from which along with the application Gronwall's lemma we deduce the existence of a positive constant $C = C(t, R)$ such that

$$\mathbb{E} |\mathbf{w}(t_R)|^2 \leq C |\xi_1 - \xi_2|^2.$$

The proof of the lemma is now finished. \square

Now we continue with the proof of Theorem 5.1

Proof of Theorem 5.1. Owing to the Theorem 4.1 and the fact that $\tilde{\eta}(A \times [0, t])$, $A \times [0, t] \in \mathcal{B}(Z \times \mathbb{R}_+)$ is time homogeneous, the Markovian property of $\mathbf{u}(t; \xi)$, $\xi \in H$, can be checked using the same argument as in [24] (see also [2]). Now we want to check that $\mathcal{P}_t(C_b(H)) \subset C_b(H)$. For this purpose let us consider $\xi \in H$ and a sequence $\{\xi_m : m \in \mathbb{N}\} \subset H$ such that $\xi_m \rightarrow \xi$ as $m \rightarrow \infty$. Let us prove that

$$\mathcal{P}_t \phi(\xi_m) \rightarrow \mathcal{P}_t \phi(\xi), \forall \phi \in C_b(H),$$

as m tends to infinity. To shorten notation we set $\tau_R = \tau_R^{\xi_m} \wedge \tau_R^\xi$ where the stopping times τ_R^ξ is defined as in (71). For any $t \in [0, T]$, $T \geq 0$ and $\phi \in C_b(H)$, we have

$$\begin{aligned} |\mathcal{P}_t \phi(\xi_m) - \mathcal{P}_t \phi(\xi)| &= \left| \mathbb{E} \left(\left[\phi(\mathbf{u}(t; \xi_m)) - \phi(\mathbf{u}(t; \xi)) \right] \mathbf{1}_{[t < \tau_R] \cup [t \geq \tau_R]} \right) \right| \\ &\leq \left| \mathbb{E} \left(\left[\phi(\mathbf{u}(t; \xi_m)) - \phi(\mathbf{u}(t; \xi)) \right] \left(\mathbf{1}_{[t \geq \tau_R^{\xi_m}]} + \mathbf{1}_{[t \geq \tau_R^\xi]} \right) \right) \right| \\ &\quad + \left| \mathbb{E} \left(\left[\phi(\mathbf{u}(t; \xi_m)) - \phi(\mathbf{u}(t; \xi)) \right] \mathbf{1}_{[t < \tau_R]} \right) \right|. \end{aligned}$$

Thanks to the fact that $\mathbb{E} |\mathbf{u}(t; \xi)|^2 < C(\xi)$, $\forall \xi \in H$ (see the estimate in Theorem 3.2), we obtain that for any $\varepsilon > 0$ there exists m_1 such that for any $R > m_1$

$$\mathbb{P}(\tau_R^{\xi_m} \geq t) + \mathbb{P}(\tau_R^\xi \geq t) \leq \frac{\varepsilon}{4\|\phi\|_\infty},$$

where

$$\|\phi\|_\infty = \sup_{\xi \in H} |\phi(x)|.$$

Thus

$$|\mathcal{P}_t \phi(\xi_m) - \mathcal{P}_t \phi(\xi)| \leq \left| \mathbb{E} \left(\left[\phi(\mathbf{u}(t; \xi_m)) - \phi(\mathbf{u}(t; \xi)) \right] \mathbf{1}_{[t < \tau_R]} \right) \right| + 2\|\phi\|_\infty \frac{\varepsilon}{4\|\phi\|_\infty}.$$

That is,

$$|\mathcal{P}_t \phi(\xi_m) - \mathcal{P}_t \phi(\xi)| \leq \left| \mathbb{E} \left(\left[\phi(\mathbf{u}(t; \xi_m)) - \phi(\mathbf{u}(t; \xi)) \right] \mathbf{1}_{[t < \tau_R]} \right) \right| + \frac{\varepsilon}{2}.$$

Since $\mathbf{1}_{[t < \tau_R]} \leq 1$ and $t \wedge \tau_R = t$ when $t < \tau_R$, we readily have that

$$|\mathcal{P}_t \phi(\xi_m) - \mathcal{P}_t \phi(\xi)| \leq \left| \mathbb{E} \left(\left[\phi(\mathbf{u}(t_R; \xi_m)) - \phi(\mathbf{u}(t_R; \xi)) \right] \right) \right| + \frac{\varepsilon}{2}, \quad (74)$$

where we have put $t_R = t \wedge \tau_R$. By the continuity of ϕ , for the same $\varepsilon > 0$ as above we can find $\kappa > 0$ such that if $|\mathbf{u}(t_R; \xi_m) - \mathbf{u}(t_R; \xi)| < \kappa$ we have

$$|\phi(\mathbf{u}(t_R; \xi_m)) - \phi(\mathbf{u}(t_R; \xi))| < \frac{\varepsilon}{4}. \quad (75)$$

Note that from (74) we derive that

$$\begin{aligned} |\mathcal{P}_t \phi(\xi_m) - \mathcal{P}_t \phi(\xi)| &\leq \left| \mathbb{E} \left(\left[\phi(\mathbf{u}(t_R; \xi_m)) - \phi(\mathbf{u}(t_R; \xi)) \right] \mathbf{1}_{\{|\mathbf{u}(t_R; \xi_m) - \mathbf{u}(t_R; \xi)| \geq \kappa\}} \right) \right| \\ &\quad + \left| \mathbb{E} \left(\left[\phi(\mathbf{u}(t_R; \xi_m)) - \phi(\mathbf{u}(t_R; \xi)) \right] \mathbf{1}_{\{|\mathbf{u}(t_R; \xi_m) - \mathbf{u}(t_R; \xi)| < \kappa\}} \right) \right| + \frac{\varepsilon}{2}, \end{aligned}$$

from which all together with (75) we derive that

$$\begin{aligned} |\mathcal{P}_t \phi(\xi_m) - \mathcal{P}_t \phi(\xi)| &\leq 2\|\phi\|_\infty \mathbb{P} \left(|\mathbf{u}(t_R; \xi_m) - \mathbf{u}(t_R; \xi)| \geq \kappa \right) \\ &\quad + \left| \mathbb{E} \left(\left[\phi(\mathbf{u}(t_R; \xi_m)) - \phi(\mathbf{u}(t_R; \xi)) \right] \mathbf{1}_{\{|\mathbf{u}(t_R; \xi_m) - \mathbf{u}(t_R; \xi)| < \kappa\}} \right) \right| + \frac{\varepsilon}{2}. \end{aligned} \quad (76)$$

Invoking the estimate (72) and Chebychev's inequality we obtain that

$$2\|\phi\|_\infty \mathbb{P} \left(|\mathbf{u}(t_R; \xi_m) - \mathbf{u}(t_R; \xi)| \geq \kappa \right) \leq \frac{2\|\phi\|_\infty C}{\kappa^2} |\xi_m - \xi|^2. \quad (77)$$

But as $\xi_m \rightarrow \xi$ as $m \rightarrow \infty$ we have that for any $\delta > 0$ there exists $m_2 > 0$ such that if $m > m_2$ we have $|\xi_m - \xi|^2 < \delta$. Choosing $\delta = \frac{\varepsilon \kappa^2}{8C\|\phi\|_\infty}$ we can derive from (77) that

$$2\|\phi\|_\infty \mathbb{P} \left(|\mathbf{u}(t_R; \xi_m) - \mathbf{u}(t_R; \xi)| \geq \kappa \right) \leq \frac{\varepsilon}{4}. \quad (78)$$

So combining (75), (76) and (78) we see that for any $\varepsilon > 0$ there exists $m_0 > 0$ such that if $m > m_0$ then

$$|\mathcal{P}_t \phi(\xi_m) - \mathcal{P}_t \phi(\xi)| < \varepsilon,$$

which shows that \mathcal{P}_t is a Fellerian semigroup. \square

Owing to Theorem 5.1 we can discuss about the existence of the invariant measure associated to the semigroup \mathcal{P}_t .

Theorem 5.3. *The Markovian semigroup \mathcal{P}_t has at least one invariant measure μ . Moreover, μ is concentrated on V , i.e. $\mu(V) = 1$.*

Proof. Let $\{T_n; n \in \mathbb{N}\} \subset [0, \infty)$ be a sequence such that $T_n \nearrow \infty$ as $n \rightarrow \infty$. For any $A \in \mathcal{B}(H)$ let us set

$$\mu_n(A) = \frac{1}{T_n} \int_0^{T_n} \mathbb{P}(\mathbf{u}(t; \xi) \in A) dt.$$

It is clear that μ_n defines a measure on $(H, \mathcal{B}(H))$. Let $R > 0$ and $A_R = \{\mathbf{u} : \|\mathbf{u}\|_2 > R\}$. Using Chebychev's inequality and Fubini's Theorem we see that

$$\mu_n(A_R) \leq \frac{1}{R^2} \frac{1}{T_n} \mathbb{E} \int_0^{T_n} \|\mathbf{u}(s; \xi)\|_2^2 ds.$$

Owing to the estimate in Theorem 3.2 we have that

$$\mu_n(A_R) \leq \frac{C(1 + |\xi|^2)}{R^2}.$$

This implies that $\mu_n(A_R) \rightarrow 0$ uniformly in n as $R \rightarrow \infty$. Since the ball $B_R = V \setminus A_R$ is compact in H , we conclude that the family of measures μ_n is tight on H . This yields that there exists a subsequence μ_{n_k} and a measure μ defined on $(H, \mathcal{B}(H))$ such that

$$\int_H \phi(x) \mu_{n_k}(dx) \rightarrow \int_H \phi(x) \mu(dx), \forall \phi \in C_b(H).$$

Since \mathcal{P}_t satisfies the Markov-Feller property, we can infer from Krylov-Bogoluibov's theorem that it admits an invariant measure which is equal to μ .

It remains to show that μ is concentrated on V . For this purpose it is sufficient to show that $\mu(H \setminus V) = 0$. To do so we will first show that

$$\mu_n(H \setminus V) = 0, \forall n.$$

Thanks to the estimate in Theorem 3.2 we can find a set $I \times \Omega_0 \subset \Omega_{T_n}, T_n \geq 0$ ($\Omega_{T_n} = [0, T_n] \times \Omega$) with $\lambda \otimes \mathbb{P}(\Omega_t \setminus I \times \Omega_0) = 0$ and $\mathbf{u}(t; \xi)(\omega) \in V$ for any $(t, \omega) \in I \times \Omega_0$. This fact implies that

$$\mathbb{P}\left(\int_0^{T_n} \mathbf{1}_N(t, \omega) dt\right) = 0,$$

where

$$N = \{(t, \omega) \in \Omega_{T_n} : \mathbf{u}(t; \xi)(\omega) \in H \setminus V\}.$$

Owing to Fubini's theorem we infer the existence of $J \subset [0, T_n]$ with $\lambda([0, T_n] \setminus J) = 0$ and

$$\mathbb{P}(\{\omega \in \Omega : \mathbf{u}(t; \xi) \in H \setminus V\}) = 0,$$

for any $t \in J$. Setting $N_t = \{\omega \in \Omega : \mathbf{u}(t; \xi) \in H \setminus V\}$ for any $t \in J$, we find that

$$\begin{aligned} \mu_n(H \setminus V) &= \frac{1}{T_n} \int_0^{T_n} \mathbb{P}(N_t) dt, \\ &= \frac{1}{T_n} \int_0^{T_n} \mathbf{1}_J(t) \mathbb{P}(N_t) dt, \\ &= 0. \end{aligned}$$

This means that the support of μ_n is included in V . Since μ is the weak limit of μ_n , we derive from [20, Theorem 2.2] that the support of μ is included in V . \square

Our next concern is to check whether the invariant measure μ is ergodic or not. In fact we will find that it is ergodic provided that κ_1 is large enough. We will make our claim clearer later on, but for now let us prove an important fact about the invariant measure μ .

Proposition 5.4. *If $2\kappa_1\lambda_1^2 - \ell_1 > 0$, then there exists a constant $\tilde{L} > 0$ depending only on $\kappa_1, \lambda_1, \ell_0, \ell_1$ such that*

$$\int_H (|\xi|^2 + \|\xi\|_2^2) \mu(dx) < \tilde{L}. \quad (79)$$

Proof. First we should notice that by Itô's formula we have

$$\begin{aligned} |\mathbf{u}(t; \xi)|^2 + 2\kappa_1 \int_0^t \|\mathbf{u}(s; \xi)\|_2^2 ds + 2 \int_0^t \langle \mathcal{A}_p \mathbf{u}(s; \xi), \mathbf{u}(s; \xi) \rangle ds \\ = |\xi|^2 + \int_0^t \int_Z |\sigma(s, \mathbf{u}(s; \xi), z)|^2 \eta(dz, ds) + 2 \int_0^t \int_Z (\sigma(s, \mathbf{u}(s; \xi), z), \mathbf{u}(s-; \xi)) \tilde{\eta}(dz, ds). \end{aligned} \quad (80)$$

Now for any $\varepsilon > 0$ let $\Phi(y) = \frac{y}{1+\varepsilon y}, y \in \mathbb{R}_+$. It is clear that

$$\begin{aligned} \Phi'(y) &= \frac{1}{(1+\varepsilon y)^2}, \\ \Phi''(y) &= \frac{-2\varepsilon}{(1+\varepsilon y)^3}, \end{aligned}$$

for any $y \geq 0$. It is clear from the last equality that $\Phi''(y) < 0$, and $|\Phi''| \leq 2\varepsilon$ for any $y \geq 0$. Notice also that $\eta(dz, ds) = \tilde{\eta}(dz, ds) + \nu(dz)ds$ and

$$|\sigma(s, \mathbf{u}(s, \xi), z)|^2 + 2(\sigma(s, \mathbf{u}(s, \xi), z), \mathbf{u}(s-; \xi)) = |\sigma(s, \mathbf{u}(s, \xi), z) + \mathbf{u}(s-; \xi)|^2 - |\mathbf{u}(s-; \xi)|^2.$$

By setting $Y(t) = |\mathbf{u}(t; \xi)|^2$ and $\Psi = |\sigma(s, \mathbf{u}(s, \xi), z) + \mathbf{u}(s-; \xi)|^2 - |\mathbf{u}(s-; \xi)|^2$ we can rewrite (80) in the following form

$$\begin{aligned} Y(t) + 2\kappa_1 \int_0^t \|\mathbf{u}(s; \xi)\|_2^2 ds + 2 \int_0^t \langle \mathcal{A}_p \mathbf{u}(s; \xi), \mathbf{u}(s; \xi) \rangle ds &= |\xi|^2 + \int_0^t \int_Z |\sigma(s, \mathbf{u}(s; \xi), z)|^2 \nu(dz) ds \\ &\quad + \int_0^t \int_Z \Psi \tilde{\eta}(dz, ds). \end{aligned}$$

Applying Itô's formula to $\Phi(Y)$ we obtain that

$$\begin{aligned} \Phi(Y(t)) + 2\kappa_1 \int_0^t \Phi'(Y(s)) \|\mathbf{u}(s; \xi)\|_2^2 ds + 2 \int_0^t \Phi'(Y(s)) \langle \mathcal{A}_p \mathbf{u}(s; \xi), \mathbf{u}(s; \xi) \rangle ds \\ = \Phi(|\xi|^2) + \int_0^t \int_Z \left(\Phi(Y(s-)) + \Psi - \Phi(Y(s-)) - \Phi'(Y(s-))\Psi \right) \eta(dz, ds) \\ + \int_0^t \Phi'(Y(s)) \int_Z |\sigma(s, \mathbf{u}(s; \xi), z)|^2 \nu(dz) ds + \int_0^t \int_Z \left(\Phi(Y(s-)) + \Psi - \Phi(Y(s-)) \right) \tilde{\eta}(dz, ds). \end{aligned}$$

Since $\langle \mathcal{A}_p \mathbf{u}(s; \xi), \mathbf{u}(s; \xi) \rangle \geq 0$ and $\Phi'(y) > 0$ for any $y \geq 0$, we can drop out the third term from the left-hand side of the last equation. Therefore we obtain that

$$\begin{aligned} \Phi(Y(t)) + 2\kappa_1 \int_0^t \Phi'(Y(s)) \|\mathbf{u}(s; \xi)\|_2^2 ds &\leq \Phi(|\xi|^2) + \int_0^t \Phi'(Y(s)) \left(\int_Z |\sigma(s, \mathbf{u}(s; \xi), z)|^2 \nu(dz) \right) ds \\ &\quad + \int_0^t \int_Z \left(\int_0^1 \Phi''(Y(s-)) + \theta \Psi \right) \tilde{\eta}(dz, ds) + \int_0^t \int_Z \left(\int_0^1 \Phi''(Y(s-)) + \theta \Psi \right) \nu(dz) ds \\ &\quad + \int_0^t \int_Z \left(\int_0^1 \Phi''(Y(s-)) + \theta \Psi \right) \tilde{\eta}(dz, ds), \end{aligned}$$

where we have used the identities

$$\begin{aligned} \Phi(y + \psi) - \Phi(y) &= \int_0^1 \Phi'(y + \theta\psi) \psi d\theta, \\ \Phi(y + \psi) - \Phi(y) - \Phi'(y)\psi &= \int_0^1 \Phi''(y + \theta\psi) \psi^2 d\theta. \end{aligned}$$

Since $|\Phi'(\cdot)| < 1$ and $|\Phi''| < 2\varepsilon$ and

$$\mathbb{E}\Psi^r \leq C\mathbb{E}(1 + |\mathbf{u}(s; \xi)|^{2r}) < C,$$

with $r = 1, 2$, the stochastic integrals

$$\begin{aligned} \int_0^t \int_Z \left(\int_0^1 \Phi'(Y(s)) + \theta \Psi \right) \tilde{\eta}(dz, ds), \\ \int_0^t \int_Z \left(\int_0^1 \Phi''(Y(s-)) + \theta \Psi \right) \tilde{\eta}(dz, ds), \end{aligned}$$

are martingales with zero mean. Hence taking the mathematical expectation yields

$$\begin{aligned} \mathbb{E}\Phi(Y(t) - \Phi(|\xi|^2)) &\leq \mathbb{E} \int_0^t \Phi'(Y(s)) \left(\int_Z |\sigma(s, \mathbf{u}(s; \xi), z)|^2 \nu(dz) \right) ds \\ &\quad + \mathbb{E} \int_0^t \int_Z \left(\int_0^1 \Phi''(Y(s-)) + \theta \Psi \right) \nu(dz) ds \\ &\quad - 2\kappa_1 \mathbb{E} \int_0^t \Phi'(Y(s)) \|\mathbf{u}(s; \xi)\|_2^2 ds. \end{aligned} \tag{81}$$

Since

$$\begin{aligned}\Phi''(Y(s-) + \theta\Psi)\Psi^2 &= \frac{-2\varepsilon\Psi^2}{(1 + \varepsilon Y(s) + \varepsilon\theta\Psi)^3} \\ &= \frac{-2\varepsilon\Psi^2}{(1 + \varepsilon\theta|\sigma(s, \mathbf{u}(s, \xi), z) + \mathbf{u}(s-; \xi)|^2 - \varepsilon(1 - \theta)|\mathbf{u}(s-; \xi)|^2)},\end{aligned}$$

we see that $\Phi''(Y(s-) + \theta\Psi)\Psi^2 \leq 0$ for any $\theta \in [0, 1]$. Therefore we can drop out the second term in the right-hand side of (81), use item (2) in Condition 1 to obtain

$$\begin{aligned}\mathbb{E}\Phi(Y(t)) + 2\kappa_1\mathbb{E}\int_0^t \Phi'(Y(s))\|\mathbf{u}(s; \xi)\|_2^2 ds &\leq \Phi(|\xi|^2) + \ell_1\mathbb{E}\int_0^t \Phi'(Y(s))|\mathbf{u}(s; \xi)|^2 ds \\ &\quad + \ell_0\mathbb{E}\int_0^t \Phi'(Y(s))ds.\end{aligned}\tag{82}$$

By using Poincaré's inequality (see (2)) the last estimate becomes

$$\begin{aligned}\mathbb{E}\Phi(Y(t)) + 2\kappa_1\lambda_1^2\mathbb{E}\int_0^t \Phi'(Y(s))|\mathbf{u}(s; \xi)|_2^2 ds &\leq \Phi(|\xi|^2) + \ell_1\mathbb{E}\int_0^t \Phi'(Y(s))|\mathbf{u}(s; \xi)|^2 ds \\ &\quad + \ell_0\mathbb{E}\int_0^t \Phi'(Y(s))ds.\end{aligned}\tag{83}$$

By integrating both side of this last inequality wrt μ on H and using the fact that

$$\int_H \mathbb{E}\phi(\mathbf{u}(s, \xi))\mu(dx) = \int_H \phi(\xi)\mu(dx), \forall \phi \in C_b(H), \quad (\mu \text{ is an invariant measure})\tag{84}$$

we obtain from (83) that

$$(2\kappa_1\lambda_1^2 - \ell_1) \int_H \frac{|\xi|^2}{(1 + \varepsilon|\xi|^2)^2} \mu(dx) \leq \ell_0 \int_H \frac{1}{(1 + \varepsilon|\xi|^2)^2} \mu(dx).$$

From this inequality we obtain that

$$\int_H \frac{|\xi|^2}{(1 + \varepsilon|\xi|^2)^2} \mu(dx) \leq \frac{\ell_0}{2\kappa_1\lambda_1^2 - \ell_1},\tag{85}$$

where we have used the facts that $\frac{1}{(1 + \varepsilon|\xi|^2)^2} \geq 1$, $2\kappa_1\lambda_1^2 - \ell_1 > 0$ and $\mu(V) + \mu(H \setminus V) = 1$.

From (82) and (85) we derive that

$$2\kappa_1 \int_H \mathbb{E} \int_0^t \Phi'(Y(s))\|\mathbf{u}(s; \xi)\|_2^2 ds \leq \frac{\ell_0}{2\kappa_1\lambda_1^2 - \ell_1} (\ell_1 + 1) + \ell_0.\tag{86}$$

Choosing $\phi(\mathbf{u}(s; \xi)) = \int_0^t \Phi'(Y(s))\|\mathbf{u}(s; \xi)\|_2^2 ds$ and using (84) we see that

$$\int_H \frac{\|\xi\|_2^2}{(1 + \varepsilon|\xi|^2)^2} \mu(dx) \leq \frac{\ell_0}{2\kappa_1(2\kappa_1\lambda_1^2 - \ell_1)} (\ell_1 + 1) + \frac{\ell_0}{2\kappa_1}.\tag{87}$$

Adding up (85) and (87) side by side, letting $\varepsilon \rightarrow 0$ and using Fatou's lemma imply that

$$\int_H (|\xi|^2 + \|\xi\|_2^2) \mu(dx) \leq \frac{\ell_0}{2\kappa_1\lambda_1^2 - \ell_1} \left(\frac{\ell_1 + 1}{2\kappa_1} + 1 \right) + \frac{\ell_0}{2\kappa_1},\tag{88}$$

which terminates the proof of the proposition. \square

We can prove the ergodicity of the invariant measure under the condition that κ_1 is large enough.

Theorem 5.5. *Assume that $2\kappa_1\lambda_1^2 > \ell_1$. Then, the Markovian semigroup \mathcal{P}_t has an invariant measure μ which is tight and ergodic on H .*

Proof. Let $\mathcal{M} \subset \mathcal{M}_1(H)$ be the set of invariant measure of \mathcal{P}_t and

$$\tilde{\ell} = \frac{\ell_0}{2\kappa_1\lambda_1^2 - \ell_1} \left(\frac{\ell_0}{2\kappa_1} + 1 \right) + \frac{\ell_0}{2\kappa_1}.$$

It is not difficult to show that \mathcal{M} is convex (see for example [28, page 296]). As before let $R > 0$ and $A_R = \{\mathbf{u} \in H : \|\mathbf{u}\|_2 > R\}$. We see from Chebychev-Markov's inequality that

$$\sup_{\mu \in \mathcal{M}} \mu(A_R) \leq \frac{1}{R^2} \int_H \|\xi\|_2^2 \mu(dx).$$

Owing to (87) we have that

$$\sup_{\mu \in \mathcal{M}} \mu(A_R) \leq \frac{\tilde{\ell}}{R^2},$$

which implies that for any $\varepsilon > 0$

$$\mu(B_V(\frac{1}{\sqrt{\varepsilon}})) \geq 1 - \varepsilon,$$

where $B_V(\frac{1}{\sqrt{\varepsilon}}) = V \setminus A_{\frac{1}{\sqrt{\varepsilon}}}$. Since $B_V(\frac{1}{\sqrt{\varepsilon}})$ is compact in H we infer that the set \mathcal{M} is tight on H . Since \mathcal{M} is non-empty, convex and tight, by Krein-Millman's theorem (see, for instance, [30, Theorem 3.65, p. 110]) it has extrema which are ergodic. We deduce from the above argument that \mathcal{P}_t has at least one invariant measure which is ergodic. \square

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